
TECHNICAL REPORT R-74

THE LATERAL RESPONSE OF AIRPLANES TO RANDOM ATMOSPHERIC TURBULENCE

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SUMMARY

The report is separated into two parts. In the first part the theoretical basis for calculating the lateral motions of an airplane flying through continuous random atmospheric turbulence is presented. The lateral motions are derived in terms of (1) the transfer functions relating the motion in the various degrees of freedom to the yawing moment, rolling moment, and side force, (2) the statistical forces and moments at the center of gravity due to gust velocities acting on the lifting surfaces of the airplane, and (3) the power spectra of the three orthogonal components of gust velocity acting on the airplane along the flight path. The method takes into account the random variations of gust velocity across the span and along the fuselage. Solutions are given in the form of equations relating the power spectra of the angular motions of the airplane to the power spectra of the gust velocities.

Three airplanes of different size are used to demonstrate the method, illustrate characteristic trends, and exhibit some simplifications possible in the calculations. For these airplanes the effects of horizontal gusts (that is, gusts parallel to the flight path) and side forces due to gusts on the airplanes were found to be negligible.

By using one of the example airplanes, a comparison is drawn between the present theory and several less comprehensive theories for calculating the effect of gusts on wings of finite span and the effect of gusts on the motions of the complete airplane.

In the second part of the report a relatively short method of calculating the lateral motions of an airplane due to random atmospheric turbulence is presented. The gust velocities are represented as equiv-

alent rigid-body rotations of the airplanes; namely, rolling gusts, yawing gusts, and side gusts. Random distributions of gust velocities across the span are taken into account in defining the rolling and yawing gusts. Complex stability derivatives are used to account for the random distribution of side gusts along the fuselage and vertical tail and the lag effect incurred as the airplane penetrates the gusts. The suggested gust spectrum is based on a simple analytical expression which fits available measurements of atmospheric turbulence. A 45-step sample calculation procedure for obtaining the response of the airplane in each degree of freedom is presented in tabular form.

INTRODUCTION

The classical theory of stability and control of airplanes has been applied to the calculation of response to controls and response to gusts. In the calculation of response to gusts, however, the angle-of-attack distributions applied to the airplane by the gust have customarily been assumed to be equivalent to those resulting from a rigid-body motion of the airplane. This method was applied in NACA Report 1 and other early reports (ref. 1) in the study of the longitudinal response of airplanes to gusts and in reference 2 to the calculation of the lateral response to gusts. Because of the method of accounting for the effect of the gusts, these analyses were restricted to the consideration of the effects of isolated gusts of some specified shape. Most early developments in the theory of response to gusts were attempts to account for factors which were important in the calculation of

¹ Supersedes NACA Technical Note 3954 by John M. Eggleston, 1957, and NACA Technical Note 4196 by John M. Eggleston and William H. Phillips, 1958.

loads. For example, the calculation of loads due to vertical gusts was extended in reference 3 to include the effects of unsteady lift and of flexibility.

In the early methods of calculating gust response, the analysis did not include such factors as the penetration effect (a result of the penetration of a given gust at different times by different parts of the airplane) and the variation of gust velocities along the fuselage and spanwise directions. In recent years, some attempts have been made to include these effects without changing the basic method of approach. In reference 4 an approximate method was introduced for estimating the penetration effect in the calculation of longitudinal response to gusts by taking into account first-order effects of the time lag between the penetration of the gust by the wing and by the tail. The method used was similar to that introduced in reference 5 for calculating the effect of lag of downwash. A similar application of this concept to the calculation of lateral response to gusts was presented in reference 6.

A more realistic treatment of random disturbances in the atmosphere was made possible with the introduction of a method of calculating the statistical response of an airplane in terms of the statistical properties of atmospheric gust velocities. This method, known as statistical dynamics, was first applied in reference 7 to the measurement of the longitudinal response of an airplane to random vertical gusts and has since been used extensively in longitudinal gust loads studies.

A theoretical analysis for the statistical lateral response of an airplane to gusts was given in reference 8 where, in order to approximate the effects of the span, the turbulence was considered to be represented by a combination of rolling gusts and side gusts having arbitrary distributions along the flight path. With this representation of turbulence, however, the question arose as to the relative magnitudes of the rolling and side gusts which would satisfactorily represent atmospheric turbulence. In reference 8 the assumption was made that an arbitrary gust distribution across the span could be represented by a constant antisymmetric gradient which was assumed to be equal to the gradients existing in the corresponding distribution of side gusts. Experimental flight measurements of reference 9 indicated, however, that the measured gradients of

vertical gusts across the span did not agree with the theoretical calculations of reference 8, and the assumption of nonisotropic atmospheric turbulence was employed in order to fit the theory to the flight-test results.

A more realistic method of utilizing statistical dynamics to take into account random gusts both along the flight path and across the span was reported in reference 10 in outlining a means of calculating the longitudinal response of an airplane. The analysis was consistent with isotropic turbulence and required only the statistical gust distributions measured at one point in the turbulence.

Therefore, with the advent of statistical dynamics and more realistic means of analyzing the random distributions over the airplane as well as along the flight path, the original methods of treating gusts no longer appear to be justified. While it would be desirable to continue to treat the effects of gusts as equivalent to certain rigid-body motions of the airplane, such treatment must be modified in order to be consistent with statistical gust inputs. The forces and moments on the airplane rather than the resulting motions are statistically related to the gust input; that is, a random distribution of gusts across the span and along the fuselage of an airplane will produce changes in forces and moments which are related to the gust velocities along the flight path only in a statistical manner. The purpose of part I of this report is to establish the statistical relations between gust velocities and the forces and moments affecting the lateral motions of the airplane and to use these forces and moments to compute the lateral motion.

In preparation for the calculations of part I, two preceding papers have been published. One of these papers (ref. 11) contains a method for calculating the side force and yawing moment on the fuselage and tail of an airplane in continuous gusts. The other (ref. 12) presents the lateral forces and moments on a wing due to three-dimensional random turbulence for a number of wing load distributions and for wings of various spans. This work (ref. 12) is an extension of the theory given in reference 10 for defining the lift on a wing due to random turbulence.

Part I presents a method of utilizing the statistical forces and moments derived in references 11 and 12 for the calculation of the lateral motion of an airplane in any lateral degree of freedom due

to atmospheric turbulence. First, the governing equations are derived and the power spectral solutions of the motions are given in terms of linear airplane transfer functions, statistical force and moment coefficients, and power spectra of the three orthogonal components of atmospheric gust velocity. General expressions are given or referenced for each of these independent functions. Next, solutions for three airplanes are shown with trends and possible simplifications noted. Finally, a comparison is made between the results of the present theory and those of earlier theories, both by comparing the gust distributions required to produce equivalent airplane motion and by comparing the airplane motions for the same assumed gust characteristics.

In part II, it is shown that the more conventional method of assuming gust-velocity distributions equivalent to rigid-body motions of the airplane may be refined to provide results equivalent to those given by the method of part I. The refinements consist in replacing some of the constant aerodynamic stability derivatives with complex quantities to account for the gust penetration effects and determining the correct relations between the spectra of rolling, yawing, and side gusts to yield results in agreement with the more exact analysis. The method of part II requires somewhat simpler calculations than the method of part I and provides a clearer physical picture of the relations between the various sources of lateral gust disturbances.

The reader who is not interested in the detailed development of the theory may go directly to the second part of the report, which affords a short method valid for most aircraft. A brief review of the background material and a detailed explanation of the method are given therein in a self-contained form.

SYMBOLS

A	constant used in references 6, 8, and 9 to indicate relative amplitudes of power spectra
b	wing span
C.P.	cross power
D	nondimensional operator, $\frac{b}{U} \frac{d}{dt}$
$E(w_g)$	plot of Φ_{C_l} of reference 12
F'	Fourier transform of a quantity

f	arbitrary function of time
G	matrix containing stability derivatives relating airplane moments and forces to gust velocities
\tilde{G}	alternate form of G containing frequency-dependent stability derivatives
h_p	pressure altitude
h_z	height of center of pressure of vertical tail above X -axis
I	imaginary part of complex quantity
$i = \sqrt{-1}$	arbitrary constant
K	nondimensional radius of gyration about X -axis, $\frac{k_x}{b}$
K_X	nondimensional radius of gyration about Z -axis, $\frac{k_z}{b}$
K_Z	nondimensional product of inertia, $\frac{k_{xz}}{b^2}$
K_{xz}	radii of gyration
$k' = \omega I_z / U$	product of gyration
$k_n = \omega x_n / U$	integral scale of turbulence
k_x, k_z	tail length measured from center of gravity to center of pressure of vertical tail
k_{xz}	arbitrary terms which may take on values 1, 2, 3 . . .
L	rolling velocity, $\dot{\phi}$
l_i	dynamic pressure, $\frac{1}{2} \rho U^2$
m, n	real part of complex quantity
p	yawing velocity, $\dot{\psi}$
q	wing area
R	profile height (see eq. (44))
r	time interval over which power spectrum is evaluated (eq. (83))
S	time
s	relative velocity between airplane and general air mass
T	velocity along X -axis
t	velocity along Y -axis
U	weight of airplane
u	velocity along Z -axis
v	three orthogonal axes of airplane (see fig. 1)
W	coordinates with reference to X -, Y -, and Z -axis
w	coefficient of drag at zero lift
X, Y, Z	
x, y, z	
$C_{D,0}$	

C_L	lift coefficient, $\frac{\text{Lift}}{qS}$	ω	circular frequency
C_l	rolling-moment coefficient, $\frac{\text{Rolling moment}}{qSb}$	ω'	reduced frequency, $\omega b/U$
C_n	yawing-moment coefficient, $\frac{\text{Yawing moment}}{qSb}$	ω'_n	natural reduced frequency
C_Y	side-force coefficient, $\frac{\text{Side force}}{qS}$	$ $	absolute value of a quantity (determinant of a matrix)
α	angle of attack	$[]$	rectangular matrix
α_0	trim (steady state) angle of attack	$\{\}$	column matrix
β	angle of sideslip	$[\]$	row matrix
$\beta' = b/L$			
Γ	effective dihedral angle		
γ	flight-path angle		
Δ	matrix of airplane equations of motion in still air		
θ	angle of pitch		
λ	wavelength, $\frac{2\pi U}{\omega}$		
μ	relative density factor, $\frac{\text{Mass}}{\rho S b}$		
ρ	density of atmosphere		
σ	sidewash angle		
Φ_f	power spectral density of a function f		
$\Phi_{f_1 f_2}$	cross-power spectral density of functions f_1 and f_2		
ϕ	angle of roll		
ψ	angle of yaw		

Stability derivatives of airplane are indicated by subscript notation; for example,

$$C_{l_r} = \frac{\partial C_l}{\partial \left(\frac{rb}{2U} \right)} \quad C_{n_p} = \frac{\partial C_n}{\partial \left(\frac{pb}{2U} \right)} \quad C_{Y_\beta} = \frac{\partial C_Y}{\partial \left(\frac{r}{U} \right)}$$

Subscripts:

F	fuselage
FT	fuselage and vertical tail
T	vertical tail
W	wing
g	gust
j, k, m	generalized matrix subscripts
o	general air mass

Superscript:

- * conjugate of a complex number
- Bar over a quantity denotes mean value.
- Dot over a quantity denotes derivative with respect to time.

I. A THEORY FOR THE LATERAL RESPONSE OF AIRPLANES TO ATMOSPHERIC TURBULENCE

LATERAL EQUATIONS OF MOTION

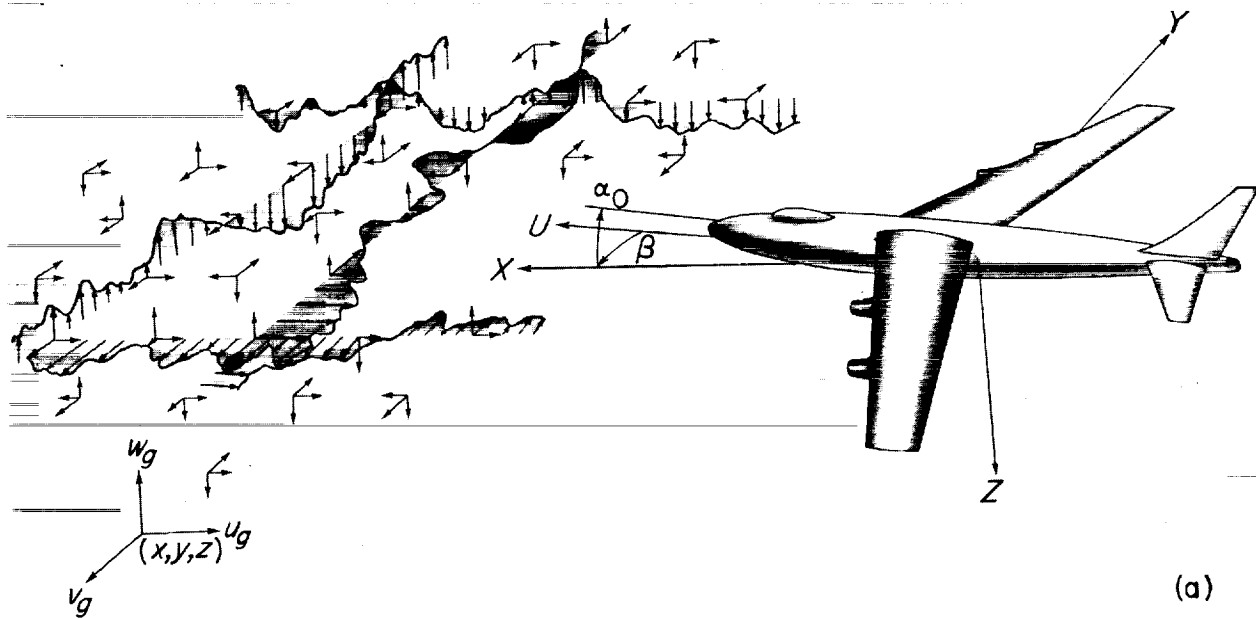
SYSTEMS OF AXES AND DESCRIPTION OF TURBULENCE

The motions of an airplane flying through continuous atmospheric turbulence are defined as angular displacements, rates, or accelerations about a set of stability axes (with respect to the general air mass, the X stability axis is always aligned with the projection of the resultant mean velocity on the plane of symmetry) whose origin is fixed at the center of gravity. These stability axes have some mean orientation with respect to the body axes of the airplane denoted by the angle α_0 in the XZ -plane. The general orientation of the stability axes of the airplane with respect to the gust velocities and with respect to the body and earth axes is shown in figure 1.

At each position along the flight path of the air-

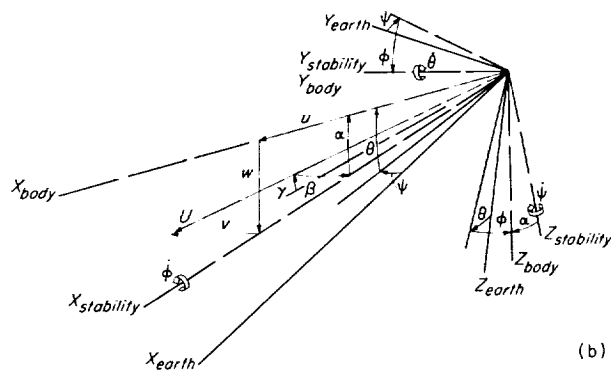
plane the instantaneous velocities of the local air mass, hereinafter designated as gusts, are instantaneously defined by three orthogonal components of velocity aligned with the stability axes of the airplane at that position. The gusts are assumed to be essentially isotropic in nature, and their statistical characteristics remain unchanged regardless of the motion or orientation of the airplane. This assumption is implicit throughout this paper.

Furthermore, the assumption is made that, during the small interval of time in which each gust acts on the airplane, the gust velocity does not change appreciably. On this basis the three components of gust velocity—aligned with, but of opposite sign to, displacements along the stability axes of the airplane—are functions only



(a) Turbulence structure and mean orientation of stability axes of airplane.

FIGURE 1.— Sign convention and stability axes of airplane flying through atmospheric turbulence.



(b) Instantaneous orientation of stability axes of airplane with respect to earth and body axes. Positive sense of velocities and angles is indicated by arrows.

FIGURE 1. — Concluded.

of their space position along the flight path. With reference to the center of gravity of the airplane, these gusts are denoted by

$$u_g = u_g(x, y, z)$$

$$v_g = v_g(x, y, z)$$

$$w_g = w_g(x, y, z)$$

The orientation of the gust axes is shown in figure 1(a).

The assumption of isotropy further relates these velocities only to the extent of

$$\overline{u_g^2} = \overline{v_g^2} = \overline{w_g^2}$$

where the bar denotes the mean value.

SYSTEM OF EQUATIONS

The lateral equations of motion of an airplane defined in the stability system of axes are generally well known. Based on these equations of motion, the frequency response of an airplane in its three lateral degrees of freedom is defined by the matrix

$$\begin{Bmatrix} \phi(\omega) \\ \psi(\omega) \\ \beta(\omega) \end{Bmatrix} = \begin{bmatrix} \frac{\phi}{C_l}(\omega) & \frac{\phi}{C_n}(\omega) & \frac{\phi}{C_r}(\omega) \\ \frac{\psi}{C_l}(\omega) & \frac{\psi}{C_n}(\omega) & \frac{\psi}{C_r}(\omega) \\ \frac{\beta}{C_l}(\omega) & \frac{\beta}{C_n}(\omega) & \frac{\beta}{C_r}(\omega) \end{bmatrix} \begin{Bmatrix} C_l(\omega) \\ C_n(\omega) \\ C_r(\omega) \end{Bmatrix} \quad (1)$$

In equation (1) the ratios ϕ/C_l , ψ/C_n , and so forth, are complex transfer functions of the airplane and relate the response of the airplane in ϕ ,

ψ , and β to a sinusoidal rolling moment, yawing moment, or side force of unit amplitude. These moments and side force are functions of the gust velocities referenced to a point (the center of gravity of the airplane). The necessary phase relations between these forces and moments and the gust velocities at the center of gravity are taken into account by expressing C_l , C_n , and C_Y as complex quantities. In terms of the gust velocities at the center of gravity, the lateral-force and moment coefficients may be expressed as follows:

$$\begin{Bmatrix} C_l(\omega) \\ C_n(\omega) \\ C_Y(\omega) \end{Bmatrix} = \begin{bmatrix} \frac{C_l}{u_g}(\omega) & \frac{C_l}{v_g}(\omega) & \frac{C_l}{w_g}(\omega) \\ \frac{C_n}{u_g}(\omega) & \frac{C_n}{v_g}(\omega) & \frac{C_n}{w_g}(\omega) \\ \frac{C_Y}{u_g}(\omega) & \frac{C_Y}{v_g}(\omega) & \frac{C_Y}{w_g}(\omega) \end{bmatrix} \begin{Bmatrix} u_g(\omega) \\ v_g(\omega) \\ w_g(\omega) \end{Bmatrix} \quad (2)$$

In equation (2) the elements of the rectangular matrix are transfer functions relating the forces and moments to the sinusoidal components of gust velocities of unit amplitude measured at the center of gravity. Inasmuch as some of these forces and moments arise as a result of gust velocities distributed over the airplane, which are, in turn, related to the gust components at the center of gravity only in a statistical sense, these transfer functions are likewise statistical in nature and are later expressed in the form of power spectra.

With the foregoing restrictions the complete relationship between the frequency response of the airplane in the three lateral motions and the gust velocities is given by the matrix equation

$$\begin{Bmatrix} \phi(\omega) \\ \psi(\omega) \\ \beta(\omega) \end{Bmatrix} = \begin{bmatrix} \frac{\phi}{C_l} & \frac{\phi}{C_n} & \frac{\phi}{C_Y} \\ \frac{\psi}{C_l} & \frac{\psi}{C_n} & \frac{\psi}{C_Y} \\ \frac{\beta}{C_l} & \frac{\beta}{C_n} & \frac{\beta}{C_Y} \end{bmatrix} \begin{bmatrix} \frac{C_l}{u_g} & \frac{C_l}{v_g} & \frac{C_l}{w_g} \\ \frac{C_n}{u_g} & \frac{C_n}{v_g} & \frac{C_n}{w_g} \\ \frac{C_Y}{u_g} & \frac{C_Y}{v_g} & \frac{C_Y}{w_g} \end{bmatrix} \begin{Bmatrix} u_g(\omega) \\ v_g(\omega) \\ w_g(\omega) \end{Bmatrix} \quad (3)$$

Thus, the moments due to the gusts and those due to the resulting motions are superposed. The equations of motion of the airplane are

assumed to be linear within the range of this resulting motion.

POWER SPECTRAL RESPONSE

Inasmuch as turbulence in the atmosphere appears to be random, the statistical response of an airplane may, in general, be predicted in terms of the power spectral densities and mean-square values of the gusts in the atmosphere. An excellent summary of this field of dynamics is presented in reference 13. From the relationship (eq. (3)) between the gust velocities and the motions of the airplane in the frequency domain, the relationship between these quantities in terms of power spectra and cross-power spectra may be readily defined.

A quantity (herein denoted, in general, by the symbol $f(t)$) which has a random variation with time in the range $-T \leq t \leq T$ and is zero elsewhere is related to its Fourier transform $F(\omega)$ by

$$F(\omega) = \frac{1}{\pi} \int_{-T}^T f(t) e^{-i\omega t} dt \quad (4)$$

where, in turn,

$$f(t) = \frac{1}{2} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \quad (5)$$

The power spectral density of the quantity is defined (see, for example, ch. 6 of ref. 14 or appendix of ref. 9) by

$$\begin{aligned} \Phi_f(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{T} F^*(\omega) F(\omega) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} |F(\omega)|^2 \end{aligned} \quad (6)$$

where asterisks are used to denote the complex conjugate of a quantity and the term $\frac{1}{T}$ indicates that the spectra are symmetrical and only positive values of ω are considered when taking the average value.

When two related quantities are measured concurrently (for example, when the yawing and rolling moments due to each gust component are measured at the same values of time), a cross-power spectral density exists between these two quantities. Where the frequency content of these

two quantities is defined by

$$\left. \begin{aligned} F_1(\omega) &= \frac{1}{\pi} \int_{-\tau}^{\tau} f_1(t) e^{-i\omega t} dt \\ F_2(\omega) &= \frac{1}{\pi} \int_{-\tau}^{\tau} f_2(t) e^{-i\omega t} dt \end{aligned} \right\} \quad (7)$$

the cross-power spectral density is defined by

$$\Phi_{f_1 f_2} = \lim_{T \rightarrow \infty} \frac{1}{T} F_1^*(\omega) F_2(\omega)$$

or

$$\begin{aligned} \Phi_{f_2 f_1} &= \lim_{T \rightarrow \infty} \frac{1}{T} F_2^*(\omega) F_1(\omega) \\ &= (\Phi_{f_1 f_2})^* \end{aligned} \quad (8)$$

The significance of this cross power lies in the fact that, if the quantities are related, a certain interchange of energy takes place between them and the amount of this interchange is defined by the cross-power spectrum.

In the special case where F_1 and F_2 are the same function and are removed in time or space (that is, $f_1(t) = f_2(t)$), equations (6) and (8) are identical and the cross power of f with itself is, in fact, the power spectrum.

These definitions may be found in most texts and in a number of papers on power spectral analysis, among them references 9, 13, and 14. On the basis of the preceding discussion, it may be shown that if each complex equation of the system defined by equation (3) is multiplied by its conjugate equation, divided through by the quantity T , and the limit as $T \rightarrow \infty$ taken on both sides of each equation, then the system of equations, by definition, is in the power spectral form. The power spectra and cross-power spectra of the variables may be recognized and so denoted, the spectra and cross spectra of the gust velocities being unknown variables. An example of this operation is shown in appendix A.

By using this or any other suitable method of power spectral analysis, the system of equation (1)

may be defined in terms of power and cross-power spectral densities such that

$$\begin{Bmatrix} \Phi_\phi \\ \Phi_\psi \\ \Phi_\beta \end{Bmatrix} = \begin{bmatrix} \left| \frac{\phi}{C_l} \right|^2 & \left| \frac{\phi}{C_n} \right|^2 & \left| \frac{\phi}{C_Y} \right|^2 \\ \left| \frac{\psi}{C_l} \right|^2 & \left| \frac{\psi}{C_n} \right|^2 & \left| \frac{\psi}{C_Y} \right|^2 \\ \left| \frac{\beta}{C_l} \right|^2 & \left| \frac{\beta}{C_n} \right|^2 & \left| \frac{\beta}{C_Y} \right|^2 \end{bmatrix} \begin{Bmatrix} \Phi_{C_l} \\ \Phi_{C_n} \\ \Phi_{C_Y} \end{Bmatrix} + 2R \begin{bmatrix} \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} & \left(\frac{\phi}{C_n} \right)^* \frac{\phi}{C_Y} & \left(\frac{\phi}{C_Y} \right)^* \frac{\phi}{C_l} \\ \left(\frac{\psi}{C_l} \right)^* \frac{\psi}{C_n} & \left(\frac{\psi}{C_n} \right)^* \frac{\psi}{C_Y} & \left(\frac{\psi}{C_Y} \right)^* \frac{\psi}{C_l} \\ \left(\frac{\beta}{C_l} \right)^* \frac{\beta}{C_n} & \left(\frac{\beta}{C_n} \right)^* \frac{\beta}{C_Y} & \left(\frac{\beta}{C_Y} \right)^* \frac{\beta}{C_l} \end{bmatrix} \begin{Bmatrix} \Phi_{C_l C_n} \\ \Phi_{C_n C_Y} \\ \Phi_{C_Y C_l} \end{Bmatrix} \quad (9)$$

where $2R$ means twice the real part of the product of the two matrices.

On the right-hand side of equation (9), the product of the first two matrices is the contribution of the power spectra of the forces and moments, and the product of the second two matrices is the contribution of the cross-power spectra between the moments and forces produced by any one gust component and also the cross powers between the forces and moments produced by any two gust components. The powers and cross powers of the forces and moments may, in turn, be related to the powers and cross powers of the gust velocities. In so doing, a great simplification is obtained when atmospheric turbulence is considered to be isotropic within the frequency range affecting

the response of the airplane.

By virtue of the assumption of isotropic homogeneous turbulence, the phasing between the u_g and w_g components and the v_g and w_g components is unrelated and the cross power of the components is zero. The phasing between the two gust components lying in the XY -plane of the airplane, that is, u_g and v_g , is unrelated when these velocities are measured at a given spanwise station (for example, along the airplane center line) but is related when u_g at one spanwise station is compared with v_g at another spanwise station. However, the effects of the cross power between these two components are assumed to be insignificant, and justification for this assumption is made in a subsequent section.

Thus,

$$\Phi_{u_g r_g}(\omega) = \Phi_{r_g w_g}(\omega) = \Phi_{w_g u_g}(\omega) = 0 \quad (10)$$

By the method of multiplying by the conjugate and taking the limits, the matrix relationship of equation (2) is defined in the form of power spectra by

$$\begin{Bmatrix} \Phi_{C_l} \\ \Phi_{C_n} \\ \Phi_{C_Y} \end{Bmatrix} = \begin{bmatrix} \left| \frac{C_l}{u_g} \right|^2 & \left| \frac{C_l}{v_g} \right|^2 & \left| \frac{C_l}{w_g} \right|^2 \\ \left| \frac{C_n}{u_g} \right|^2 & \left| \frac{C_n}{v_g} \right|^2 & \left| \frac{C_n}{w_g} \right|^2 \\ \left| \frac{C_Y}{u_g} \right|^2 & \left| \frac{C_Y}{v_g} \right|^2 & \left| \frac{C_Y}{w_g} \right|^2 \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \quad (11)$$

Likewise by cross-multiplying the equations in the matrix of equation (2) to conform with the definition of a cross-power spectrum, the cross spectra between the forces and moments are

given by the matrix

$$\begin{Bmatrix} \Phi_{C_l C_n} \\ \Phi_{C_n C_Y} \\ \Phi_{C_Y C_l} \end{Bmatrix} = \begin{bmatrix} \left(\frac{C_l}{u_g} \right)^* \frac{C_n}{u_g} & \left(\frac{C_l}{v_g} \right)^* \frac{C_n}{v_g} & \left(\frac{C_l}{w_g} \right)^* \frac{C_n}{w_g} \\ \left(\frac{C_n}{u_g} \right)^* \frac{C_Y}{u_g} & \left(\frac{C_n}{v_g} \right)^* \frac{C_Y}{v_g} & \left(\frac{C_n}{w_g} \right)^* \frac{C_Y}{w_g} \\ \left(\frac{C_Y}{u_g} \right)^* \frac{C_l}{u_g} & \left(\frac{C_Y}{v_g} \right)^* \frac{C_l}{v_g} & \left(\frac{C_Y}{w_g} \right)^* \frac{C_l}{w_g} \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \quad (12)$$

A proof of the relationships of equations (11) and (12) is given in appendix B.

CORRELATION BETWEEN AIRPLANE COMPONENTS

Generally, a force or moment acting on or about the center of gravity of the airplane is due to the penetration of the gust by each component of the airplane, and the correlation or phasing between the development of these forces and moments must be included. Complete generalization to all possible combinations of components would lead to lengthy and unnecessary expansions; however, all

lifting surfaces which might reasonably be expected to contribute significantly to the lateral motion of the airplane as well as the components of the gust velocity that will significantly influence the forces and moments on these lifting surfaces are incorporated. For some unusual configuration where other lifting surfaces are thought to be important, the method of including these factors is clearly indicated.

The force and moment coefficients which include only those functions deemed pertinent to the lateral motion of the airplane are defined by

$$\left. \begin{aligned} C_l(u_g, v_g, w_g) &= (C_l)_W(u_g, v_g, w_g) + (C_l)_T(v_g) \\ C_n(u_g, v_g, w_g) &= (C_n)_W(u_g, w_g) + (C_n)_{FT}(v_g) \\ C_Y(u_g, v_g, w_g) &= (C_Y)_{FT}(v_g) \end{aligned} \right\} \quad (13)$$

In terms of the complex force and moment derivatives, the force and moment matrix is thus defined by

$$\begin{Bmatrix} (C_l)_W \\ (C_l)_T \\ (C_n)_W \\ (C_n)_{FT} \\ (C_Y)_{FT} \end{Bmatrix} = \begin{bmatrix} \left(\frac{C_l}{u_g} \right)_W & \left(\frac{C_l}{v_g} \right)_W & \left(\frac{C_l}{w_g} \right)_W \\ 0 & \left(\frac{C_l}{v_g} \right)_T & 0 \\ \left(\frac{C_n}{u_g} \right)_W & 0 & \left(\frac{C_n}{w_g} \right)_W \\ 0 & \left(\frac{C_n}{v_g} \right)_{FT} & 0 \\ 0 & \left(\frac{C_Y}{v_g} \right)_{FT} & 0 \end{bmatrix} \begin{Bmatrix} u_g \\ v_g \\ w_g \end{Bmatrix} \quad (14)$$

By the same method as was applied to equation (2) to obtain equations (11) and (12) (multiplying by the conjugate and taking the limits), the power spectra and cross-power spectra of the force and moment relationships of equation (14) are immediately definable as

$$\begin{Bmatrix} \Phi_{(C_l)_W} \\ \Phi_{(C_l)_T} \\ \Phi_{(C_n)_W} \\ \Phi_{(C_n)_{FT}} \\ \Phi_{(C_Y)_{FT}} \end{Bmatrix} = \begin{bmatrix} \left| \frac{C_l}{u_g} \right|_W^2 & \left| \frac{C_l}{v_g} \right|_W^2 & \left| \frac{C_l}{w_g} \right|_W^2 \\ 0 & \left| \frac{C_l}{v_g} \right|_T^2 & 0 \\ \left| \frac{C_n}{u_g} \right|_W^2 & 0 & \left| \frac{C_n}{w_g} \right|_W^2 \\ 0 & \left| \frac{C_n}{v_g} \right|_{FT}^2 & 0 \\ 0 & \left| \frac{C_Y}{v_g} \right|_{FT}^2 & 0 \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \quad (15)$$

$$\begin{Bmatrix} \Phi_{(C_l)_W(C_l)_T} \\ \Phi_{(C_l)_W(C_n)_W} \\ \Phi_{(C_l)_W(C_n)_{FT}} \\ \Phi_{(C_l)_W(C_Y)_{FT}} \\ \Phi_{(C_l)_T(C_n)_W} \\ \Phi_{(C_l)_T(C_n)_{FT}} \\ \Phi_{(C_l)_T(C_Y)_{FT}} \\ \Phi_{(C_n)_W(C_n)_{FT}} \\ \Phi_{(C_n)_W(C_Y)_{FT}} \\ \Phi_{(C_n)_{FT}(C_Y)_{FT}} \end{Bmatrix} = \begin{bmatrix} 0 & \left(\frac{C_l}{v_g} \right)_W^* \left(\frac{C_l}{v_g} \right)_T & 0 \\ \left(\frac{C_l}{u_g} \right)_W^* \left(\frac{C_n}{u_g} \right)_W & 0 & \left(\frac{C_l}{w_g} \right)_W^* \left(\frac{C_n}{w_g} \right)_W \\ 0 & \left(\frac{C_l}{v_g} \right)_W^* \left(\frac{C_n}{v_g} \right)_{FT} & 0 \\ 0 & \left(\frac{C_l}{v_g} \right)_W^* \left(\frac{C_Y}{v_g} \right)_{FT} & 0 \\ 0 & 0 & 0 \\ 0 & \left(\frac{C_l}{v_g} \right)_T^* \left(\frac{C_n}{v_g} \right)_{FT} & 0 \\ 0 & \left(\frac{C_l}{v_g} \right)_T^* \left(\frac{C_Y}{v_g} \right)_{FT} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \left(\frac{C_n}{v_g} \right)_{FT}^* \left(\frac{C_Y}{v_g} \right)_{FT} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \quad (16)$$

The complete power-spectral-density relationships including the lateral moments and forces due to the various lifting surfaces of the airplane are similar to those given in equation (9) with the quantities defined in equations (15) and (16) inserted. When the left-hand side of equation (15) is denoted by $\{\Phi_{C_j}\}$ and the left-hand side of equation (16), by $\{\Phi_{C_j C_k}\}_{j \neq k}$, the complete re-

lationship is defined by

$$\{\Phi_{C_i}\} = \left| \frac{\phi_i}{C_j} \right|^2 \{\Phi_{C_j}\} + 2\text{R} \left[\left(\frac{\phi_i}{C_j} \right)^* \frac{\phi_i}{C_k} \right] \{\Phi_{C_j C_k}\}_{j \neq k}$$

where ϕ_1 is ϕ , ϕ_2 is ψ , and so forth. The complex force and moment derivatives contained in equation (14) were derived in such a manner that the phase relationships between the moments and the gust disturbances are given with reference to

gusts acting at the center of gravity. For this reason, the transfer functions $\frac{\phi_i}{C_j}$ are those given in equation (1). As an illustration, the total response of the airplane in roll is obtained in the following form:

$$\begin{aligned} \Phi_\phi = & \left| \frac{\phi}{C_l} \right|^2 \{ \Phi_{(C_l)W} + \Phi_{(C_l)T} \} + \left| \frac{\phi}{C_Y} \right|^2 \Phi_{(C_Y)FT} \\ & + \left| \frac{\phi}{C_n} \right|^2 \{ \Phi_{(C_n)W} + \Phi_{(C_n)FT} \} \\ & + 2R \left[\left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_l} \Phi_{(C_l)W(C_l)T} + \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \Phi_{(C_l)W(C_n)W} \right. \\ & + \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \Phi_{(C_l)W(C_n)FT} + \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_Y} \Phi_{(C_l)W(C_Y)FT} \\ & + \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \Phi_{(C_l)T(C_n)FT} + \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_Y} \Phi_{(C_l)T(C_Y)FT} \\ & \left. + \left(\frac{\phi}{C_n} \right)^* \frac{\phi}{C_Y} \Phi_{(C_n)FT(C_Y)FT} + 3 \text{ zero terms} \right] \quad (17) \end{aligned}$$

Other degrees of freedom may be obtained directly by replacing ϕ with the appropriate symbol, such as ψ or β .

By substituting in equation (17) the expressions for the power and cross-power terms of the force and moments from equations (15) and (16), the motion is defined in terms of the gust velocities:

$$\Phi_\phi = \left| \frac{\phi}{u_g} \right|^2 \Phi_{u_g} + \left| \frac{\phi}{v_g} \right|^2 \Phi_{v_g} + \left| \frac{\phi}{w_g} \right|^2 \Phi_{w_g} \quad (18)$$

where

$$\begin{aligned} \left| \frac{\phi}{u_g} \right|^2 = & \left| \frac{\phi}{C_l} \right|^2 \left| \frac{\partial C_l}{\partial u_g} \right|^2 + \left| \frac{\phi}{C_n} \right|^2 \left| \frac{\partial C_n}{\partial u_g} \right|^2 \\ & + 2R \left[\left(\frac{\partial C_l}{\partial u_g} \right)^* \left(\frac{\partial C_n}{\partial u_g} \right) \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \right] \quad (19) \end{aligned}$$

$$\begin{aligned} \left| \frac{\phi}{v_g} \right|^2 = & \left| \frac{\phi}{C_l} \right|^2 \left| \frac{\partial C_l}{\partial v_g} \right|^2 + \left| \frac{\phi}{C_l} \right|^2 \left| \frac{\partial C_l}{\partial v_g} \right|^2 + \left| \frac{\phi}{C_n} \right|^2 \left| \frac{\partial C_n}{\partial v_g} \right|^2 + \left| \frac{\phi}{C_Y} \right|^2 \left| \frac{\partial C_Y}{\partial v_g} \right|^2 \\ & + 2R \left\{ \left(\frac{\partial C_l}{\partial v_g} \right)^* \left(\frac{\partial C_l}{\partial v_g} \right) \left| \frac{\phi}{C_l} \right|^2 \right. \\ & + \left(\frac{\partial C_n}{\partial v_g} \right)^* \left(\frac{\partial C_Y}{\partial v_g} \right) \left(\frac{\phi}{C_n} \right)^* \frac{\phi}{C_Y} + \left[\left(\frac{\partial C_l}{\partial v_g} \right)^* \left(\frac{\partial C_n}{\partial v_g} \right) + \left(\frac{\partial C_l}{\partial v_g} \right)^* \left(\frac{\partial C_n}{\partial v_g} \right) \right] \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \\ & \left. + \left[\left(\frac{\partial C_l}{\partial v_g} \right)^* \left(\frac{\partial C_Y}{\partial v_g} \right) + \left(\frac{\partial C_l}{\partial v_g} \right)^* \left(\frac{\partial C_Y}{\partial v_g} \right) \right] \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_Y} \right\} \quad (20) \end{aligned}$$

$$\left| \frac{\phi}{w_g} \right|^2 = \left| \frac{\phi}{C_l} \right|^2 \left| \frac{\partial C_l}{\partial w_g} \right|^2 + \left| \frac{\phi}{C_n} \right|^2 \left| \frac{\partial C_n}{\partial w_g} \right|^2 + 2R \left[\left(\frac{\partial C_l}{\partial w_g} \right)^* \left(\frac{\partial C_n}{\partial w_g} \right) \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \right] \quad (21)$$

Thus, the response of the airplane in any degree of freedom depends on the power spectra of the three gust components, the eight complex coefficients of equation (14) which relate the gust velocities to the forces and moments, and, for each degree of freedom, the three complex transfer functions relating the motion to a sinusoidal rolling moment, yawing moment, and side force of unit amplitude. These coefficients, transfer functions, and power spectra are defined in subsequent sections of this paper.

TRANSFER FUNCTIONS

The lateral equations of motion usually accepted for the rigid dynamics of an airframe for which noncoupling with the longitudinal mode and all assumptions associated with small perturbations about a trim condition are valid are given by the matrix

$$[\Delta] \begin{Bmatrix} \phi \\ \psi \\ \beta \end{Bmatrix} = \begin{Bmatrix} C_l(u_g, r_g, w_g) \\ C_n(u_g, r_g, w_g) \\ C_Y(u_g, r_g, w_g) \end{Bmatrix} \quad (22)$$

where

$$[\Delta] = \begin{bmatrix} 2\mu K_X^2 D^2 - \frac{1}{2} C_{l_p} D & -2\mu K_{xz} D^2 - \frac{1}{2} C_{l_r} D & -C_{l_\beta} \\ -2\mu K_{xz} D^2 - \frac{1}{2} C_{n_p} D & 2\mu K_z^2 D^2 - \frac{1}{2} C_{n_r} D & -C_{n_\beta} \\ -\frac{1}{2} C_{Y_p} D - C_L & \left(2\mu - \frac{1}{2} C_{Y_r}\right) D - C_L \tan \gamma & 2\mu D - C_{Y_\beta} \end{bmatrix} \quad (23)$$

The solution, in terms of the rolling moment, yawing moment, and side force may be found by methods of operational calculus. One method is used by Mokrzycki in reference 15 where, except for the sign of K_{xz} , the identical equations of motion are used but with moments from aerodynamic-control surfaces as forcing functions. Another means of solution is through matrix algebra where the solution as given by equation (1) is of the form

$$\begin{Bmatrix} \phi \\ \psi \\ \beta \end{Bmatrix} = [\Delta]^{-1} \begin{Bmatrix} C_l \\ C_n \\ C_Y \end{Bmatrix} \quad (24)$$

where

$$[\Delta]^{-1} = \frac{\text{Adjoint } [\Delta']}{|\Delta|} \quad (25)$$

The prime denotes the transpose of the matrix Δ . Such a method is found in reference 16 and other texts on matrix algebra. The solution by either method gives the complex transfer functions with real constant coefficients of the powers of D . With the common denominator denoted by $|\Delta|$ (equivalent to $D\bar{\Delta}$ of ref. 15), the transfer functions are given by

$$\frac{\phi}{C_l} = \left[4\mu^2 K_z^2 D^3 - \mu(2K_z^2 C_{Y_\beta} + C_{n_r}) D^2 + \left(\frac{1}{2} C_{n_r} C_{Y_\beta} + 2\mu C_{n_\beta} - \frac{1}{2} C_{n_\beta} C_{Y_r} \right) D - C_{n_\beta} C_L \tan \gamma \right] \frac{1}{|\Delta|} \quad (26)$$

$$\frac{\psi}{C_l} = \left[4\mu^2 K_{xz} D^3 + \mu(C_{n_p} - 2K_{xz} C_{Y_\beta}) D^2 + \frac{1}{2} (C_{n_\beta} C_{Y_p} - C_{Y_\beta} C_{n_p}) D + C_L C_{n_\beta} \right] \frac{1}{|\Delta|} \quad (27)$$

$$\begin{aligned} \frac{\beta}{C_l} = & \left\{ \mu \left[-2K_{xz} \left(2\mu - \frac{1}{2} C_{Y_r} \right) + K_z^2 C_{Y_p} \right] D^3 - \left[-2\mu C_L (K_{xz} \tan \gamma + K_z^2) \right. \right. \\ & \left. \left. + \frac{1}{2} C_{n_p} \left(2\mu - \frac{1}{2} C_{Y_r} \right) + \frac{1}{4} C_{Y_p} C_{n_r} \right] D^2 + \frac{C_L}{2} (C_{n_p} \tan \gamma - C_{n_r}) D \right\} \frac{1}{|\Delta|} \quad (28) \end{aligned}$$

$$\frac{\phi}{C_n} = \left[4\mu^2 K_{xz} D^3 + \mu(C_{l_r} - 2K_{xz} C_{Y_\beta}) D^2 - \left(\frac{1}{2} C_{l_r} C_{Y_\beta} + 2\mu C_{l_\beta} - \frac{1}{2} C_{Y_r} C_{l_\beta} \right) D + C_{l_\beta} C_L \tan \gamma \right] \frac{1}{|\Delta|} \quad (29)$$

$$\frac{\psi}{C_n} = \left[4\mu^2 K_X^2 D^3 - \mu \left(C_{l_p} + 2K_X^2 C_{Y_\beta} \right) D^2 + \frac{1}{2} (C_{l_p} C_{Y_\beta} - C_{l_\beta} C_{Y_p}) D - C_{l_\beta} C_L \right] \frac{1}{|\Delta|} \quad (30)$$

$$\frac{\beta}{C_n} = \left\{ \mu \left[K_{xz} C_{Y_p} - 2K_X^2 \left(2\mu - \frac{1}{2} C_{Y_r} \right) \right] D^3 + \left(\frac{1}{4} C_{l_r} C_{Y_p} + 2\mu K_{xz} C_L + \mu C_{l_p} \right. \right. \\ \left. \left. - \frac{1}{4} C_{l_p} C_{Y_r} + 2\mu K_X^2 C_L \tan \gamma \right) D^2 + \frac{C_L}{2} (C_{l_r} - C_{l_p} \tan \gamma) D \right\} \frac{1}{|\Delta|} \quad (31)$$

$$\frac{\phi}{C_Y} = \left[2\mu (K_Z^2 C_{l_\beta} + K_{xz} C_{n_\beta}) D^2 + \frac{1}{2} (C_{l_r} C_{n_\beta} - C_{n_r} C_{l_\beta}) D \right] \frac{1}{|\Delta|} \quad (32)$$

$$\frac{\psi}{C_Y} = \left[2\mu (K_X^2 C_{n_\beta} + K_{xz} C_{l_\beta}) D^2 + \frac{1}{2} (C_{l_\beta} C_{n_p} - C_{n_\beta} C_{l_p}) D \right] \frac{1}{|\Delta|} \quad (33)$$

$$\frac{\beta}{C_Y} = \left\{ 4\mu^2 (K_X^2 K_Z^2 - K_{xz}^2) D^4 - \mu [K_X^2 C_{n_r} + K_{xz} (C_{l_r} + C_{n_p}) + K_Z^2 C_{l_p}] D^3 + \frac{1}{4} (C_{l_p} C_{n_r} - C_{l_r} C_{n_p}) D^2 \right\} \frac{1}{|\Delta|} \quad (34)$$

$$|\Delta| = 8\mu^3 (K_X^2 K_Z^2 - K_{xz}^2) D^5 - 2\mu^2 [K_X^2 (2K_Z^2 C_{Y_\beta} + C_{n_r}) + K_Z^2 C_{l_p} + K_{xz} (C_{n_p} + C_{l_r} - 2K_{xz} C_{Y_\beta})] D^4 \\ + \mu \left[K_Z^2 (C_{Y_\beta} C_{l_p} - C_{Y_p} C_{l_\beta}) + K_X^2 (C_{Y_\beta} C_{n_r} + 4\mu C_{n_\beta} - C_{Y_r} C_{n_\beta}) + \frac{1}{2} (C_{l_p} C_{n_r} - C_{n_p} C_{l_r}) \right. \\ \left. + K_{xz} (C_{Y_\beta} C_{l_r} + C_{Y_p} C_{n_p} + 4\mu C_{l_\beta} - C_{Y_r} C_{l_\beta} - C_{Y_p} C_{n_\beta}) \right] D^3 + \left\{ -2\mu C_L [\tan \gamma (K_X^2 C_{n_\beta} + K_{xz} C_{l_\beta}) \right. \\ \left. + K_Z^2 C_{l_\beta} + K_{xz} C_{n_\beta}] + \frac{1}{4} [C_{Y_\beta} (C_{n_p} C_{l_r} - C_{l_p} C_{n_r}) + C_{l_\beta} (C_{n_r} C_{Y_p} - C_{Y_r} C_{n_p}) + C_{n_\beta} (C_{Y_r} C_{l_p} - C_{Y_p} C_{l_r})] \right. \\ \left. + \mu (C_{l_\beta} C_{n_p} - C_{n_\beta} C_{l_p}) \right\} D^2 + \frac{C_L}{2} [\tan \gamma (C_{n_\beta} C_{l_p} - C_{l_\beta} C_{n_p}) + C_{l_\beta} C_{n_r} - C_{n_\beta} C_{l_r}] D \quad (35)$$

In addition to the angular-displacement transfer functions given here, the response in terms of the derivatives of the angular displacements may be obtained by multiplying both sides of equations (26) to (34) by the operator D . For example,

$$\frac{\dot{\beta}}{C_Y} = \frac{U}{b} D \frac{\psi}{C_Y} = \frac{U}{b} \left[2\mu (K_X^2 C_{n_\beta} + K_{xz} C_{l_\beta}) D^3 \right. \\ \left. + \frac{1}{2} (C_{l_\beta} C_{n_p} - C_{n_\beta} C_{l_p}) D^2 \right] \frac{1}{|\Delta|} \quad (36)$$

For the frequency response of these transfer functions, the operator

$$D = \frac{b}{U} \frac{d}{dt} = \frac{b}{U} i\omega = i\omega' \quad (37)$$

where ω' is the reduced frequency in terms of radians per span length. Thus, the transfer functions relating the dynamics of the airframe to the lateral forces and moments are obtained from the inertial characteristics and stability derivatives of the airframe under steady flow conditions (that is, nonturbulent flow conditions).

COMPLEX MOMENT AND FORCE COEFFICIENTS

The moment and force coefficients which relate the rolling moment, yawing moment, and side force to a sinusoidal gust velocity of unit maximum amplitude acting in planes of (but of opposite sign to) the reference axes of the airplane are, in general, frequency-dependent, and only their statistical average properties are definable. Their derivation is based on generalized harmonic analysis and basic aerodynamic theory. Where the derivation is lengthy, only the results of referenced papers are given or indicated. Furthermore, only those eight coefficients necessary to equation (14) are defined. All others are considered as having small or negligible influence on the results.

Rolling moments.—Four contributions to the rolling moment on the airplane are considered. They are the rolling moment of the wing due to all three components of gust velocity and the rolling moment due to the side gust acting on the vertical tail.

In reference 12 are derived the power spectral densities of the rolling moments of a wing in isotropic turbulence due to horizontal, vertical,

and side gusts (that is, the three components of gusts considered herein). The theory considers random turbulence across the span as well as along the flight path, and the results are given for several span-loading distributions and for a range of values of the ratio of span length to the integral scale of atmospheric turbulence. The power spectra of rolling moment given in reference 12 are not repeated herein but are simply related to the notation of this paper:

$$\left| \frac{C_l}{w_g} \right|_w^2 \Phi_{u_g}(\omega) = \Phi_{C_l} \quad (\text{due to horizontal gusts, ref. 12}) \quad (38)$$

$$\left| \frac{C_l}{u_g} \right|_w^2 \Phi_{w_g}(\omega) = \Phi_{C_l} \quad (\text{due to vertical gusts, ref. 12}) \quad (39)$$

For the side gust v_g , the random variation along the flight path was assumed to be uniform across the span, an assumption that is particularly valid for airplanes having wing loading due to sideslip concentrated over a small region of the span near the plane of symmetry. Such an assumption is likewise made herein, so that

$$\left(\frac{C_l}{v_g} \right)_w = \frac{1}{U} (C_{l\beta})_w \quad (40)$$

is assumed to be independent of frequency.

The vertical-tail contribution to the rolling moment is defined by

$$\begin{aligned} (C_l)_T(\omega) &= \left(\frac{\partial C_Y}{\partial \beta} \right)_T \beta_T \frac{h_z}{qSb} e^{-i(\omega t/U)} \\ &= (C_{Y\beta})_T \frac{r_g}{U} \frac{h_z}{b} \left(1 + \frac{\partial \sigma}{\partial \beta} \right) e^{-i(\omega t/U)} \end{aligned}$$

and, hence,

$$\left(\frac{C_l}{v_g} \right)_T(\omega) = \left[(C_{Y\beta})_T \frac{h_z}{bU} \left(1 + \frac{\partial \sigma}{\partial \beta} \right) \right] e^{-i(\omega t/U)} \quad (41)$$

Thus, the rolling moment due to side gusts may be directly calculated from the known characteristics of the airframe in still air, and the rolling moment due to horizontal and vertical gusts may be obtained in power spectral form from the plots of reference 12. The fact that the real and imaginary parts may not be ascertained for the rolling moment due to u_g and w_g components is shown to be irrelevant; that is, the real and imagi-

nary parts, as such, are not necessary to the calculation of the cross-power terms involved in calculating the response of the airplane.

Yawing moments.—The yawing moments due to the u_g and w_g components of the gust are primarily due to the unsymmetrical distribution of these gust velocities across the span of the wing. Since this unsymmetrical distribution is the same characteristic of turbulence that produces the rolling moment on the wing, the yawing moment may be fairly accurately related to the rolling moment through aerodynamic relations. As given in reference 12, the expressions for the yawing moment of the wing due to small horizontal- and vertical-gust disturbances are

$$\left(\frac{C_n}{u_g} \right)_w(\omega) = \left(\frac{C_{nr}}{C_{lr}} \right)_w \left(\frac{C_l}{u_g} \right)_w(\omega) \quad (42)$$

$$\left(\frac{C_n}{w_g} \right)_w(\omega) = \left(\frac{C_{np}}{C_{lp}} \right)_w \left(\frac{C_l}{w_g} \right)_w(\omega) \quad (43)$$

where C_{nr} , C_{lr} , and so forth, are stability derivatives of the wing only at the trim (mean) angle of attack of the airplane and, as such, are not functions of frequency. Any yawing moments of the wing due to side gusts are neglected.

The yawing moment of the fuselage and vertical fin of an airplane due to penetration into side gusts v_g has been derived in reference 11 by using a simple profile shape to define the lift distribution over the fuselage and vertical tail due to sinusoidal side-gust components. As given in reference 11,

$$\begin{aligned} \left(\frac{C_n}{v_g} \right)_{FT}(\omega) &= \frac{2\pi}{bSU} \left\{ -\frac{2x_0s_0^2}{k_0^3} [(2k_0 - ik_0^2 + i2)e^{ik_0} - i2] \right. \\ &\quad + \frac{(x_2 - x_1)(s_1 - s_0)^2}{(k_2 - k_1)^3} [(2k_2 - k_1 - i2 \\ &\quad \left. - ik_1k_2 + ik_2^2)e^{-ik_2} - (k_1 - i2)e^{-ik_1}] \right\} \quad (44) \end{aligned}$$

where x_0 , x_1 , x_2 , s_0 , and s_1 are geometric dimensions of the profile shape (see sketch in appendix C of the present report) and frequency appears in the form

$$k_n = \frac{\omega x_n}{U} \quad (n=0, 1, 2)$$

For any given flight condition for which the exact value of $C_{n\beta}$ is known, it is recommended in reference 11 that small adjustments in the profile shape be made so that at zero frequency the frequency dependent parameter $\left(\frac{C_n}{v_g}\right)_{FT} (\omega=0)$ reduces

to the static stability derivative $\frac{1}{U} C_{n\beta}$.

Side force.—The side force on an airplane in gusts is considered to be primarily due to the effect of side gusts on the fuselage and vertical fin (the profile shape) of the airplane. By using a simple profile shape and slender-body theory to define the lift distribution over the fuselage and vertical tail due to sinusoidal side gusts, the side force per unit side gust as derived in reference 11 is

$$\left(\frac{C_Y}{v_g}\right)_{FT}(\omega) = \frac{2\pi}{SU} \left\{ \frac{2s_0^2}{k_0^2} \left[1 - (1 - ik_0) e^{ik_0} \right] + \left(\frac{s_1 - s_0}{k_2 - k_1} \right)^2 \left[e^{-ik_1} - (1 - ik_1 + ik_2) e^{-ik_2} \right] \right\} \quad (45)$$

where the notation is defined in the preceding section. Again it is recommended that small adjustments be made in the representative profile shape so that at zero frequency the frequency

dependent parameter $\left(\frac{C_Y}{v_g}\right)_{FT}$ reduces to the sta-

bility derivative $\frac{1}{U} C_{Y\beta}$ which is known from flight

or wind-tunnel tests for the desired flight conditions.

POWER SPECTRA OF GUST VELOCITY

As discussed in reference 17, the relationship (correlation) between gust velocities measured at two points in isotropic turbulence may be completely defined in terms of their lateral and longitudinal components (components measured perpendicular and parallel, respectively, to the vector distance between two points). As treated herein, the airplane senses only the lateral components of the v_g and w_g gusts and both the lateral and longitudinal components of the u_g gusts. (See preceding section and ref. 12.)

Both theoretical methods (ref. 17) and flight measurements (ref. 18) have shown that the following expression approximates the spectrum of

lateral components of turbulence in the atmosphere over the frequency range affecting the dynamic response of an airplane:

$$\frac{L}{\pi U} \frac{1 + 3(k')^2}{[1 + (k')^2]^2}$$

Its horizontal counterpart has the spectrum

$$\frac{L}{\pi U} \frac{2}{1 + (k')^2}$$

where $k' = \frac{\omega L}{U}$ and L is the integral scale of turbulence. The value $2\pi L$ may be considered an approximate measure of the average eddy size in the turbulence. The data of reference 18 further indicate that L is on the order of 1,000 to 2,000 feet, and probably closer to 1,000 feet.

The foregoing expressions, having asymptotic logarithmic slopes of -2 for values of $k' \rightarrow \infty$, were first suggested in reference 19 on the basis of wind-tunnel data and have since become generally accepted as reasonable expressions for defining the power spectra of lateral and longitudinal components of isotropic turbulence. Therefore, these expressions normalized by the mean-square values are used herein as the basis for calculations involving the components of atmospheric gusts as sensed at the center of gravity (point spectra):

$$\frac{\Phi_{v_g}}{\overline{v_g^2}} = \frac{\Phi_{w_g}}{\overline{w_g^2}} = \frac{L}{\pi U} \frac{1 + 3(k')^2}{[1 + (k')^2]^2} \quad (46)$$

$$\frac{\Phi_{u_g}}{\overline{u_g^2}} = \frac{L}{\pi U} \frac{2}{1 + (k')^2} \quad (47)$$

where

$$\overline{u_g^2} = \overline{v_g^2} = \overline{w_g^2}$$

$$k' = \frac{\omega L}{U}$$

$$L \approx 1,000 \text{ feet}$$

In the calculations of reference 12, which involved the gusts sensed by a wing having finite span, equations (46) and (47) were likewise used so that the data of reference 12 and this paper are in accord as to the spectra of gusts. However, equation (47), as such, is not used directly in this paper and for this reason only equation (46) is plotted herein. The variation of equation (46) with k' is shown in figure 2.

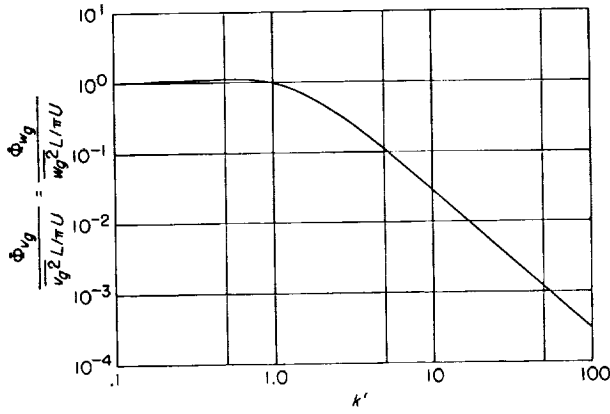


FIGURE 2.—Power spectral density of u , v , and w components of homogeneous isotropic turbulence as sensed at single point in turbulence.

APPLICATION OF METHOD

LATERAL RESPONSE OF THREE AIRPLANES

The equations which define the power spectral response of an airplane to atmospheric turbulence (eqs. (18) to (21)) have been applied to three airplanes of varying size and wing span. The flight conditions and stability derivatives of the example airplanes are given in table I, and some physical dimensions necessary to the calculation of frequency-dependent force and moment coefficients are given in table II. Some of these physical dimensions require reference 11 for their precise definition (s_0 , s_1 , x_0 , x_1 , and x_2).

By using the data presented in table I, the transfer functions of equations (26) to (35) were calculated. For discretely chosen values of reduced frequency ω' , the in-phase and out-of-phase (real and imaginary parts of the transfer function) components were evaluated with a relay computer. Care was taken to choose as one point the natural frequency of the Dutch roll mode.

By using the data of table II the complex moment and force coefficients (eqs. (38) to (45)) were evaluated at the same values of frequency as were used for evaluating the transfer functions. The relationship

$$\omega' = \frac{\omega b}{U}$$

was used to relate circular frequency in radians per second to reduced frequency in radians per span. The evaluation of certain of these coefficients requires further explanation.

The variations of $\left(\frac{C_n}{v_g}\right)_{FT}$ (eq. (44)) and $\left(\frac{C_Y}{v_g}\right)_{FT}$ (eq. (45)) with frequency are plotted in the figures of reference 11 for the example airplanes of this paper with one adjustment:

$$C_n \text{ per unit side gust (ref. 11)} = U \left(\frac{C_n}{v_g}\right)_{FT}$$

$$C_Y \text{ per unit side gust (ref. 11)} = U \left(\frac{C_Y}{v_g}\right)_{FT}$$

The designation of example airplanes for this paper and reference 11 is consistent.

The two rolling- and two yawing-moment coefficients due to u_g and w_g components acting on the wing were taken directly from the figures of reference 12. Since these four complex moment coefficients represent the spanwise integration of the power spectra of the horizontal and vertical gust components, they appear in reference 12 only in the form of the absolute value squared, whereas equations (19) and (21) appear to require their real and imaginary components. However, by substituting equation (42) into equation (19), the expression for the motion due to horizontal gusts becomes

$$\begin{aligned} \left|\frac{\phi}{u_g}\right|^2 &= \left|\frac{\phi}{C_l}\right|^2 \left|\frac{C_l}{u_g}\right|_w^2 + \left|\frac{\phi}{C_n}\right|^2 \left(\frac{C_{n_r}}{C_{l_r}}\right)_w^2 \left|\frac{C_l}{u_g}\right|_w^2 \\ &\quad + 2R \left[\left(\frac{C_l}{u_g}\right)_w^* \left(\frac{C_{n_r}}{C_{l_r}}\right)_w \left(\frac{C_l}{u_g}\right)_w \left(\frac{\phi}{C_l}\right)^* \frac{\phi}{C_n} \right] \end{aligned}$$

and since

$$z^* z = |z|^2$$

the equation becomes

$$\begin{aligned} \left|\frac{\phi}{u_g}\right|^2 &= \left\{ \left|\frac{\phi}{C_l}\right|^2 + \left|\frac{\phi}{C_n}\right|^2 \left(\frac{C_{n_r}}{C_{l_r}}\right)_w^2 \right. \\ &\quad \left. + 2R \left[\left(\frac{C_{n_r}}{C_{l_r}}\right)_w \left(\frac{\phi}{C_l}\right)^* \frac{\phi}{C_n} \right] \right\} \left|\frac{C_l}{u_g}\right|_w^2 \quad (48) \end{aligned}$$

where only the square of the absolute value of

$\left(\frac{C_l}{u_g}\right)_w$ appears and the product

$$\left|\frac{C_l}{u_g}\right|_w^2 \Phi_{u_g} \equiv \Phi_{C_l} \text{ (due to horizontal gusts)}$$

is plotted in figure 9 of reference 12.

TABLE I.—FLIGHT CONDITIONS AND STABILITY DERIVATIVES OF EXAMPLE AIRPLANES

(a) Flight conditions

Flight condition	Airplane		
	A	B	C
h_p , ft	30,000	4,000	35,000
U , ft/sec	696	318	700
W , lb	12,600	28,000	125,000
μ	50	10.9	31.83
C_L	0.242	0.177	0.443
$\tan \gamma$	0	0	0
α_0 , radians	0.0586	0.0325	0.0823

(b) Stability derivatives

Stability derivative	Airplane		
	A	B	C
Total airplane			
K_X^2	0.01485	0.0158	0.0311
K_Z^2	0.0504	0.0300	0.072
K_{XZ}	-0.00062	-0.00138	0
C_{l_p}	-0.45	-0.4875	-0.44
C_{l_r}	0.040	0.1053	0.149
C_{l_β}	-0.11	-0.1084	-0.14
C_{n_p}	-0.035	-0.0308	-0.0275
C_{n_r}	-0.15	0.1041	-0.156
C_{n_β}	0.12	0.0831	0.12
C_{Y_p}	-0.013	0	0
C_{Y_r}	0.225	0	0
C_{Y_β}	-0.58	-0.763	-0.51
C_{L_α}	4.13	5.44	5.38
Wing			
C_{n_p}/C_{l_p}	0.025	0.0208	0.054
C_{n_r}/C_{l_r}	-0.075	-0.053	-0.052
C_{l_β}	-0.0475	-0.0666	-0.0876
$C_{D,0}$	0.01	0.006	0.01
Vertical tail			
C_{Y_β}	-0.42	-0.337	-0.384
$\partial\sigma/\partial\beta$	0.19	0.242	0.215
C_{n_β}	0.183	0.129	0.154
C_{l_β}	-0.0625	-0.0418	-0.0524

TABLE II.—PHYSICAL DIMENSIONS OF EXAMPLE AIRPLANES

Dimension	Airplane		
	A	B	C
b , ft	35.25	70	116
S , sq ft	250	540	1,428
S_T , sq ft	55	71.35	230
Aspect ratio	4.98	9.07	9.4
Γ , deg	4.0	4.50	3.6
Taper ratio	0.46	0.45	0.42
h_w , ft	4.4	7.0	13.0
l_w , ft	14.8	27.6	46.6
x_0 , ft	18	15.25	48.5
x_1 , ft	5.7	22.0	36.6
x_2 , ft	20	29.4	51.6
s_0 , ft	2.7	3.0	5.3
s_1 , ft	8.5	10.6	18.5

In the parallel evaluation of the motion due to vertical gusts, the substitution of equation (43) into equation (21) yields

$$\left| \frac{\phi}{w_g} \right|^2 = \left\{ \left| \frac{\phi}{C_l} \right|^2 + \left| \frac{\phi}{C_n} \right|^2 \left(\frac{C_{n_p}}{C_{l_p}} \right)^2 + 2R \left[\left(\frac{C_{n_p}}{C_{l_p}} \right)_w \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \right] \right\} \left| \frac{C_l}{w_g} \right|^2 \quad (49)$$

where the product

$$\left| \frac{C_l}{w_g} \right|^2 \Phi_{w_g} = \Phi_{C_l} \text{ (due to vertical gusts)}$$

is plotted in figure 7 of reference 12.

Inasmuch as the figures of reference 12 are plotted against k' , the relationship

$$k' = \frac{\omega L}{U} = \frac{\omega b}{\beta' U} = \frac{\omega'}{\beta'} \quad (50)$$

where

$$\beta' = \frac{b}{L}$$

was used in the compatible choice of frequencies. Since the ratio of wing spans between the three example airplanes is on the order of

$$b_A:b_B:b_C \approx 1:2:3.3$$

the values of β' were taken as

$$\beta'_A = 0.03125$$

$$\beta'_B = 0.0625$$

$$\beta'_C = 0.103$$

which correspond to a value of L on the order of 1,100 feet. A rectangular span load distribution was used for airplane A and elliptical span load distributions were used for airplanes B and C. This choice was a matter of convenience since the plots of reference 12 indicate only a minor effect due to span loading distribution.

The power spectral responses of the airplanes were completely calculated in three degrees of freedom, ϕ , ψ , and β , by using the power spectra of the gust components given by equation (47). Angular displacements of the airplanes are shown in figures 3 to 11 as follows:

Airplane	Angular displacement	Figure number
A	ϕ	3
	ψ	4
	β	5
B	ϕ	6
	ψ	7
	β	8
C	ϕ	9
	ψ	10
	β	11

In the (a) parts of figures 3 to 11 are plotted the three components and the sum for that particular angular displacement as expressed by equation (18). As may be seen from these figures, in none of the cases considered does the u component of gust velocity contribute perceptibly to the resulting motion. Therefore, in the (b) parts of figures 3 to 11 are plotted the relative contributions to the motion of the terms due only to the v and w components of the gusts. The (b) part of each figure then is a plot of equations (20) and (49) (where eq. (49) is a modified form of eq. (21)) and, as such, does not include any power-spectrum terms. For instance, the three components

due to vertical gusts w_g shown in each (b) plot have been summed, multiplied by the power spectrum defined by equation (39), and plotted in the (a) part of each figure. The u_g and v_g contributions were treated likewise with the exception that the components due to u_g are not plotted in the (b) part of each figure.

Figures 3 to 11 are plotted in this manner to permit observation of the end results without undue complication. When the relative magnitudes of the contributing forces and moments are of particular interest, the (b) parts of the figures may be studied more closely.

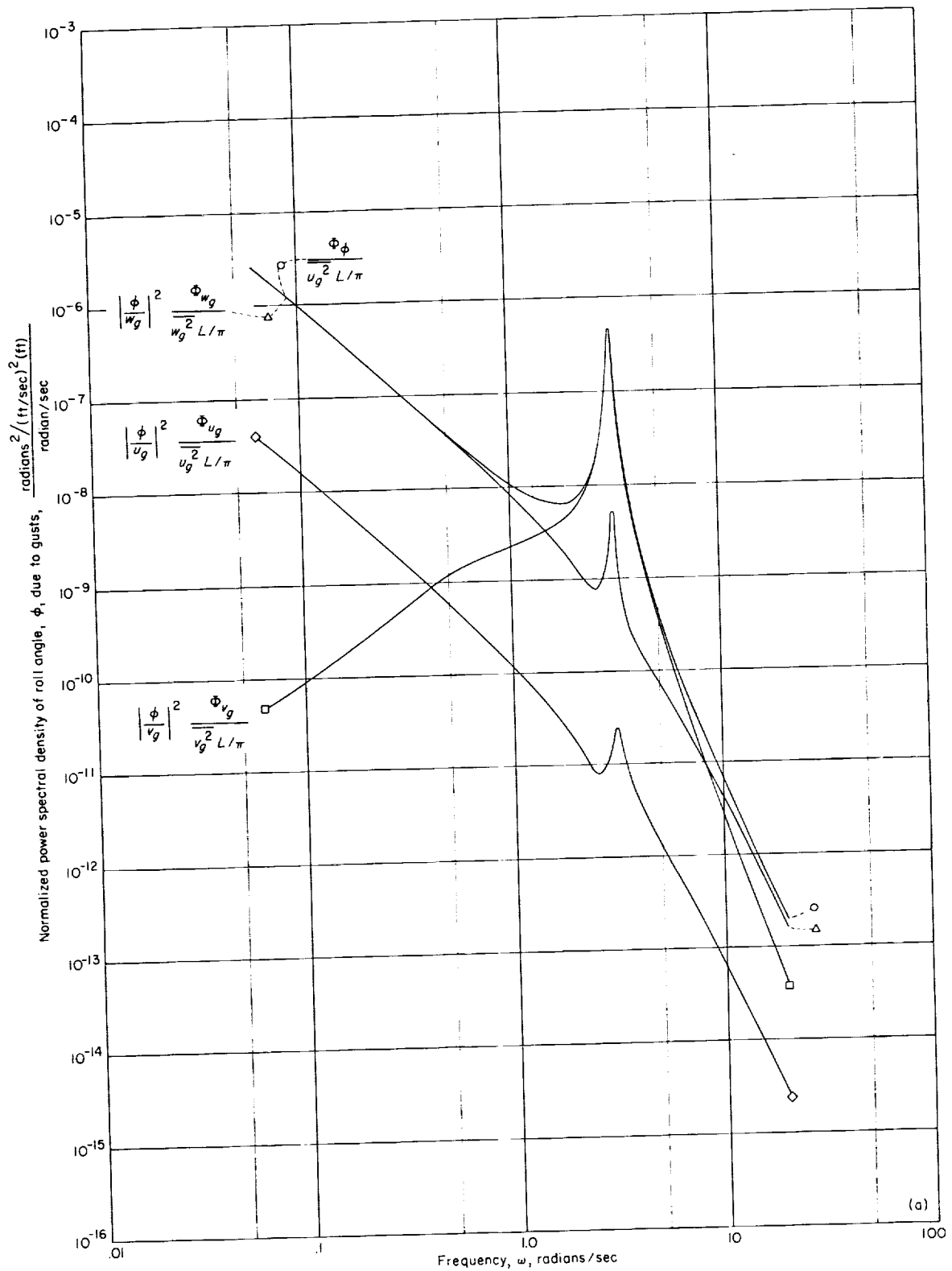
Since in order to obtain the (a) parts of these figures, the w_g components are multiplied by $\left|\frac{C_i}{w_g}\right|^2 \Phi_{u_g}$ and the v_g components multiplied by Φ_{v_g} , it should be noted that in the (b) parts of these figures the units of the w_g components are dimensionless while the v_g components have the units $\left(\frac{\text{radians}}{\text{ft/sec}}\right)^2$.

The β response plotted in figures 5, 8, and 11 is the angular displacement with respect to the general air mass (that is, still air or fixed axes) and as such does not include the displacement with respect to the local air mass (gusts).

From these example cases and figures, conclusions may be drawn on characteristics of method, trends, and further simplification of the calculations and a comparison of results from other methods of analysis may be made.

ANALYSIS

Characteristics of the method.—In applying the method of this paper, several characteristics appear to be worthy of special attention. The first of these is the importance of calculating the response of the airplane at the exact frequency where the Dutch roll mode has a maximum amplitude. Since the Dutch roll mode is generally lightly damped and the square of the amplitude is required in the calculations, the power spectral response of the angular displacements of the airplane will generally exhibit a sharp peak in this frequency range. Enough datum points should be calculated in this region to insure an accurate representation of the response.



(a) Components and sum of components of power spectral density of roll angle due to three components of gust velocity.

FIGURE 3.—Response in roll angle of airplane A flying through continuous atmospheric turbulence.

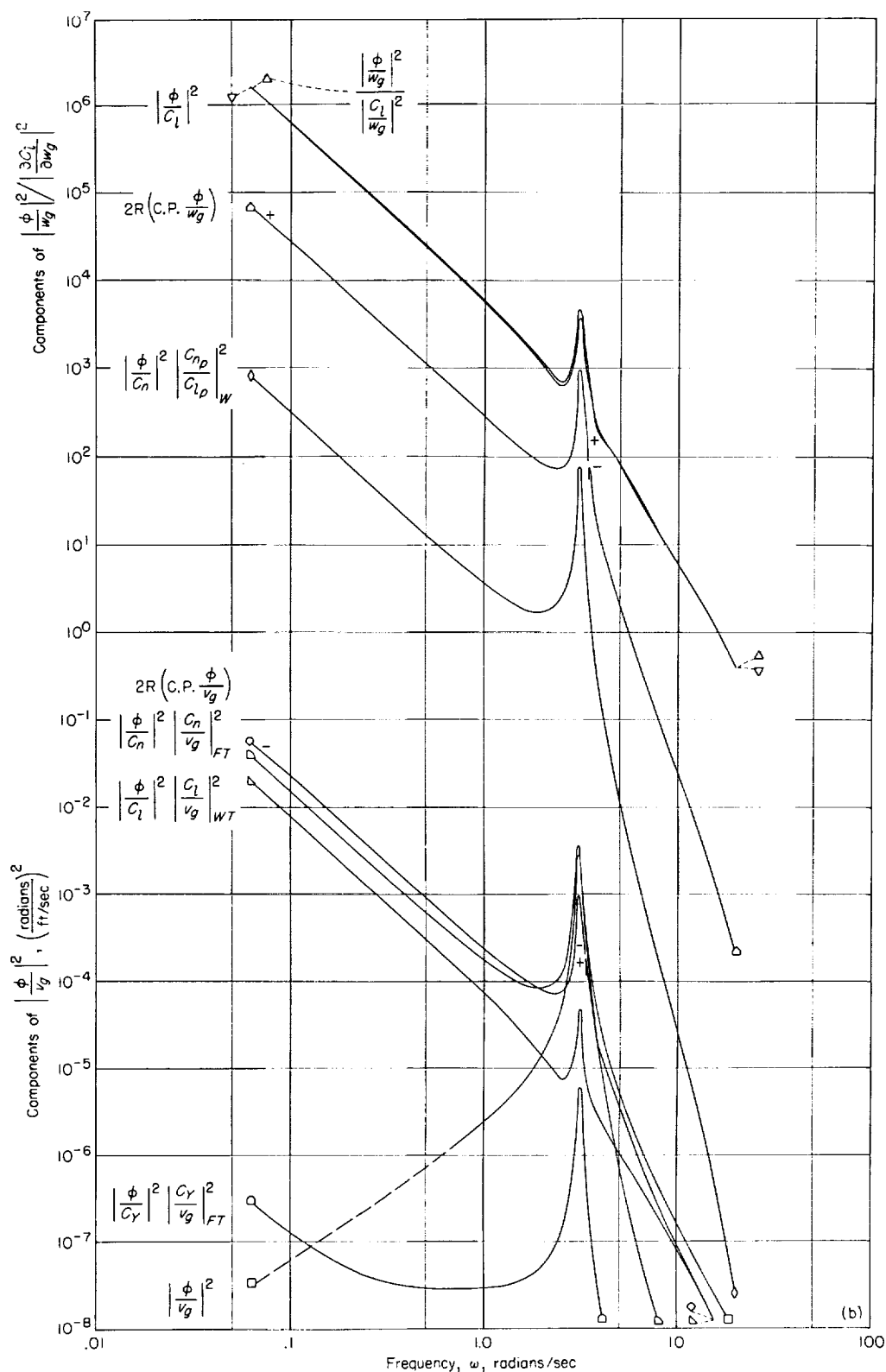
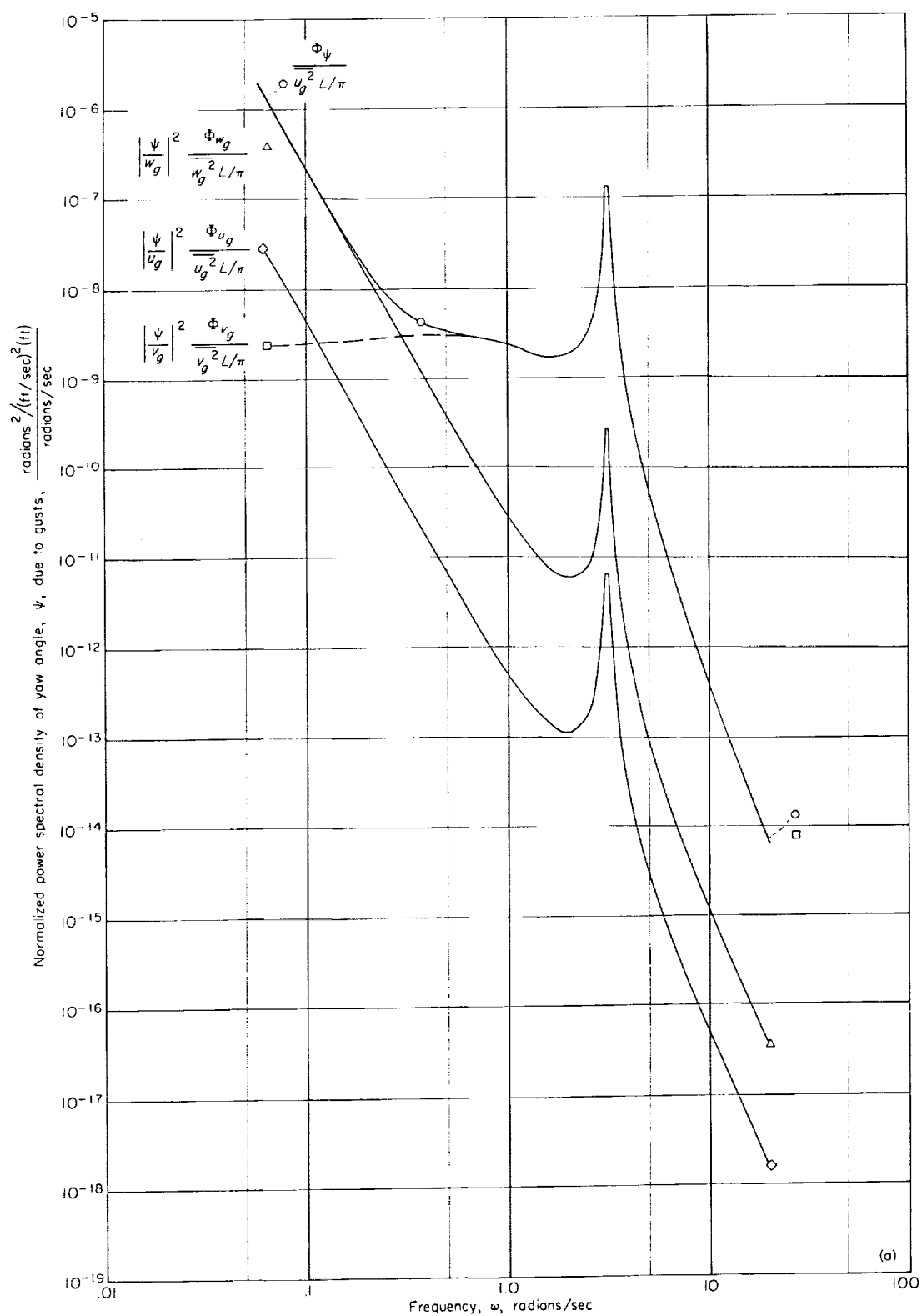
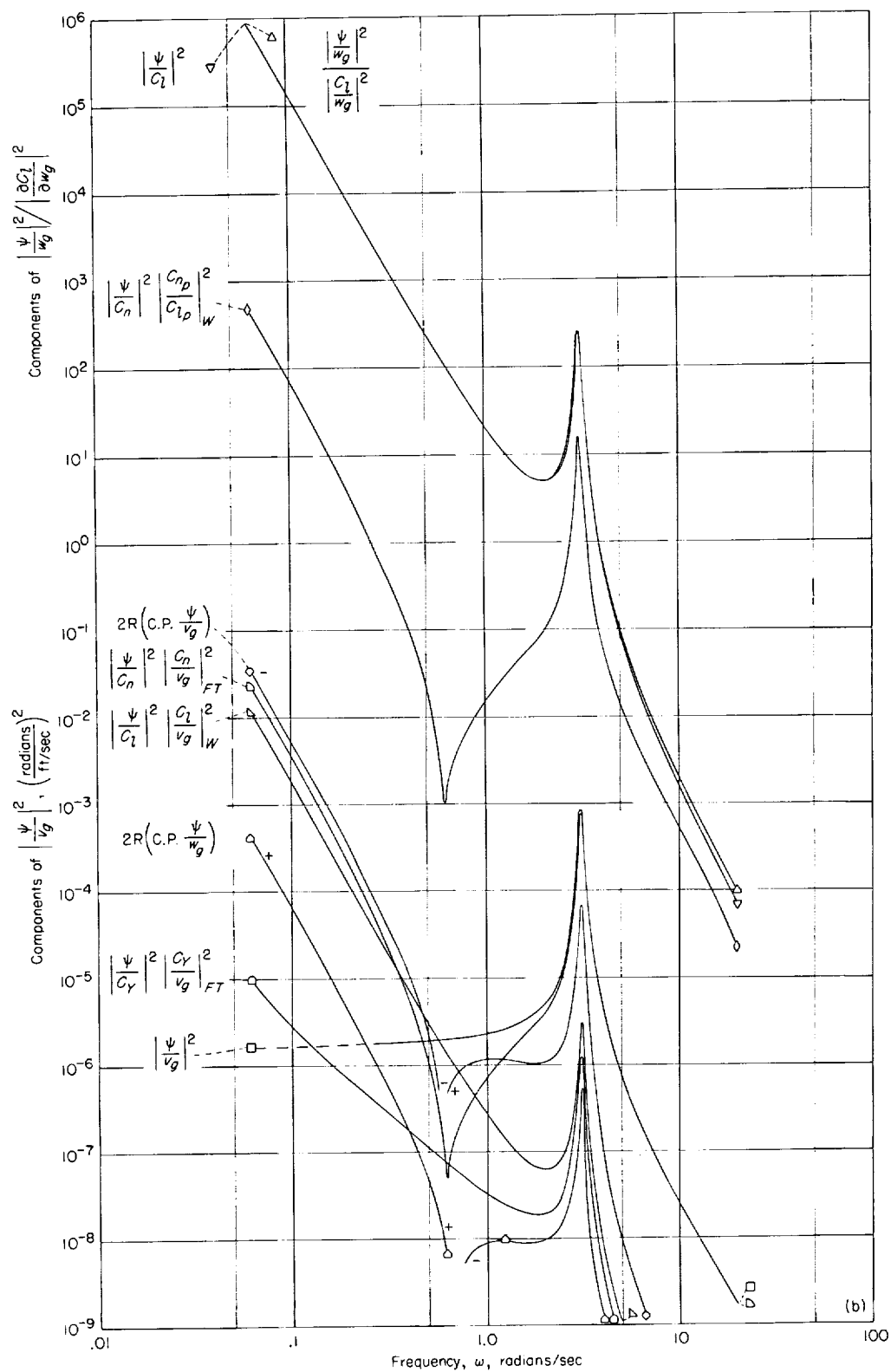


FIGURE 3.—Concluded.



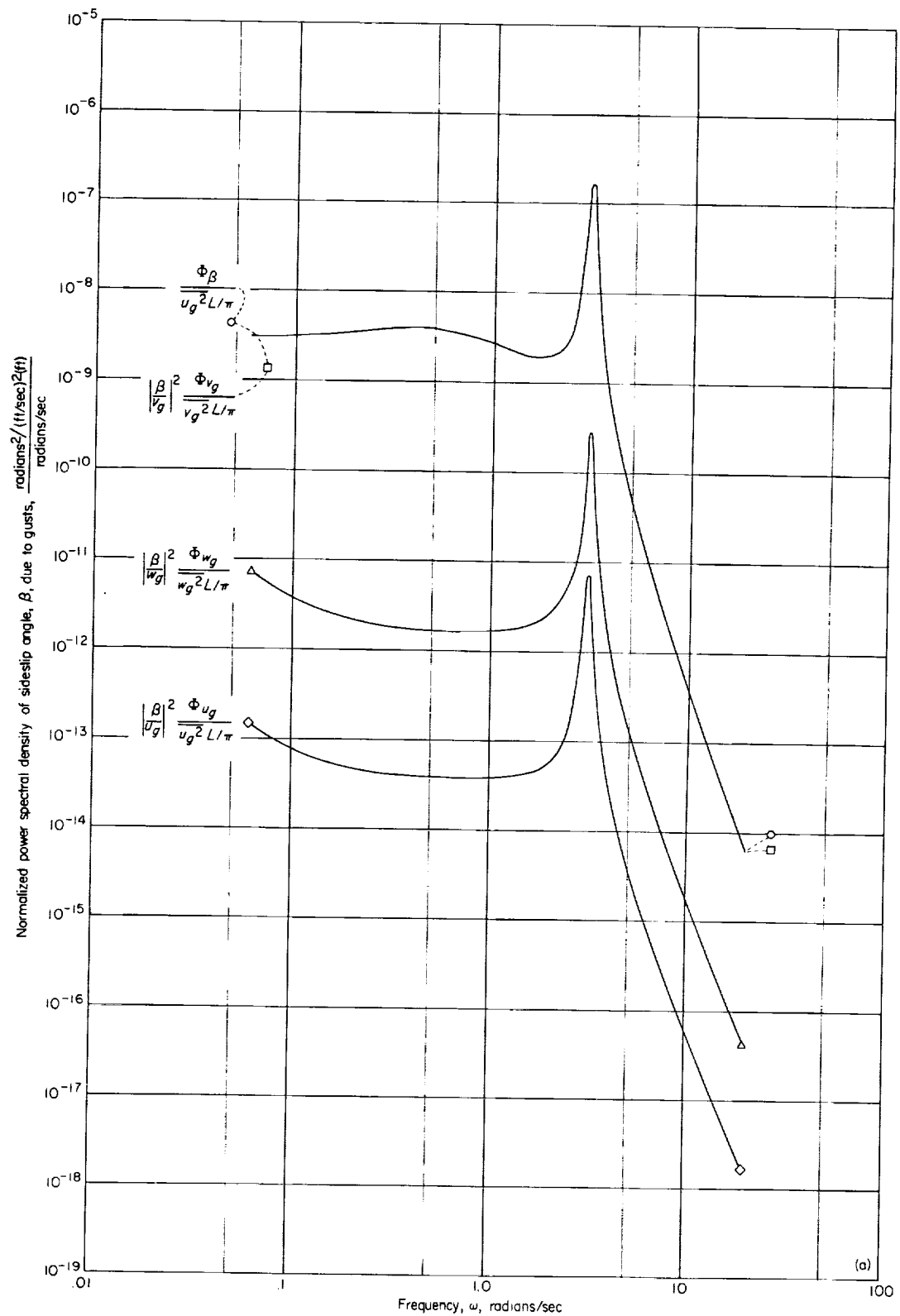
(a) Components and sum of components of power spectral density of yaw angle due to three components of gust velocity.

FIGURE 4.—Response in yaw angle of airplane A flying through continuous atmospheric turbulence.



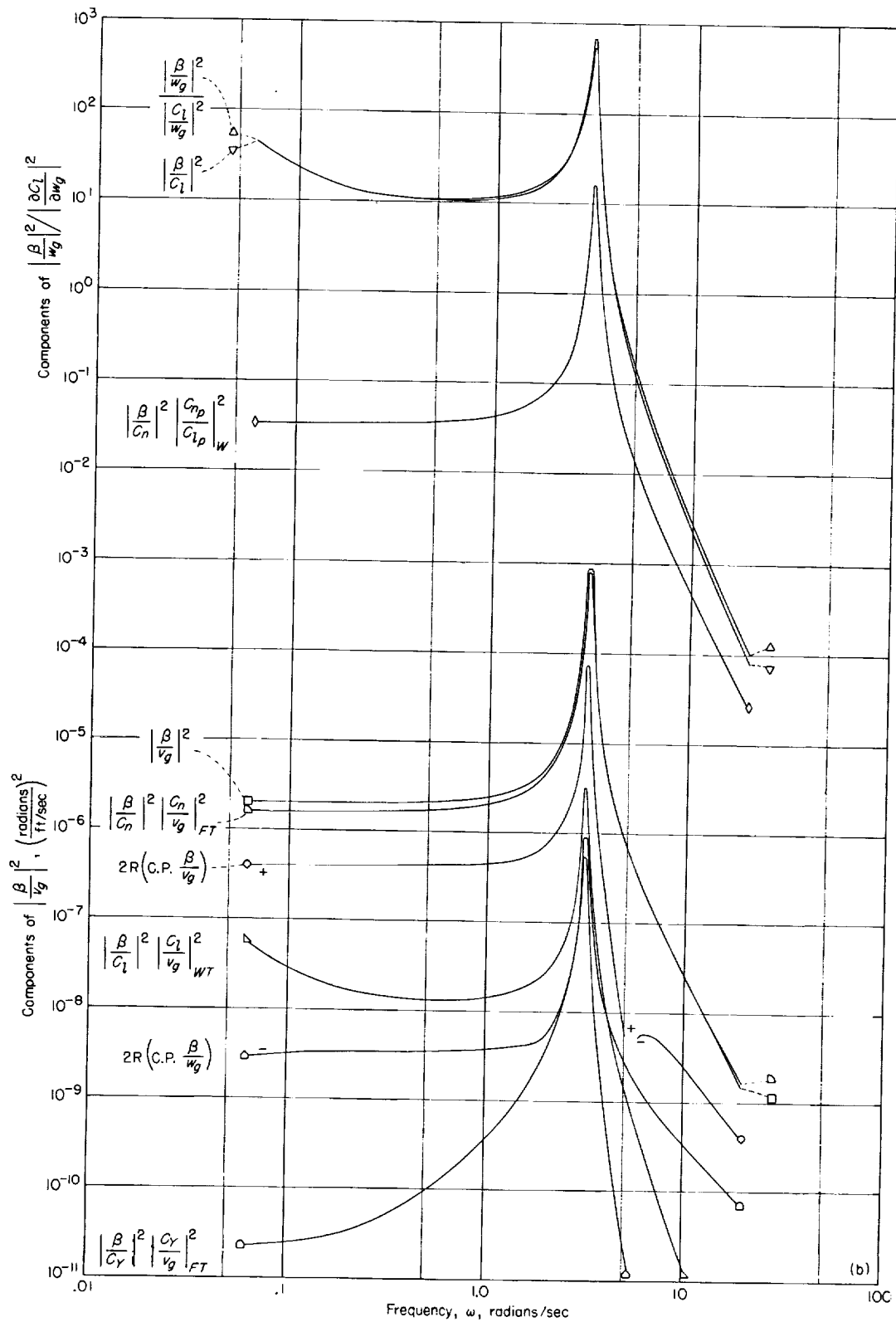
(b) Yaw angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 4.—Concluded



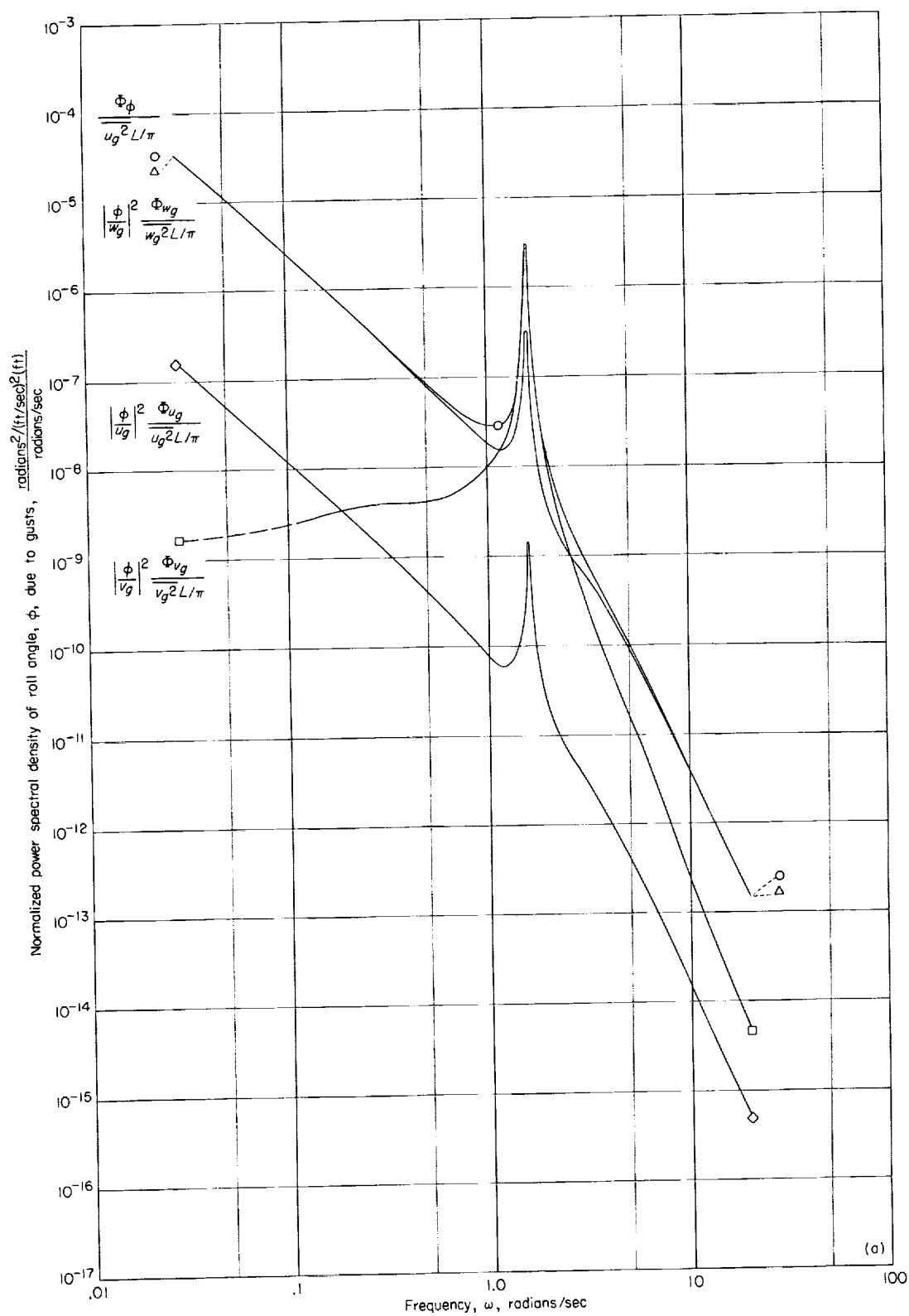
(a) Components and sum of components of power spectral density of sideslip angle due to three components of gust velocity.

FIGURE 5.—Response in sideslip angle of airplane A flying through continuous atmospheric turbulence.



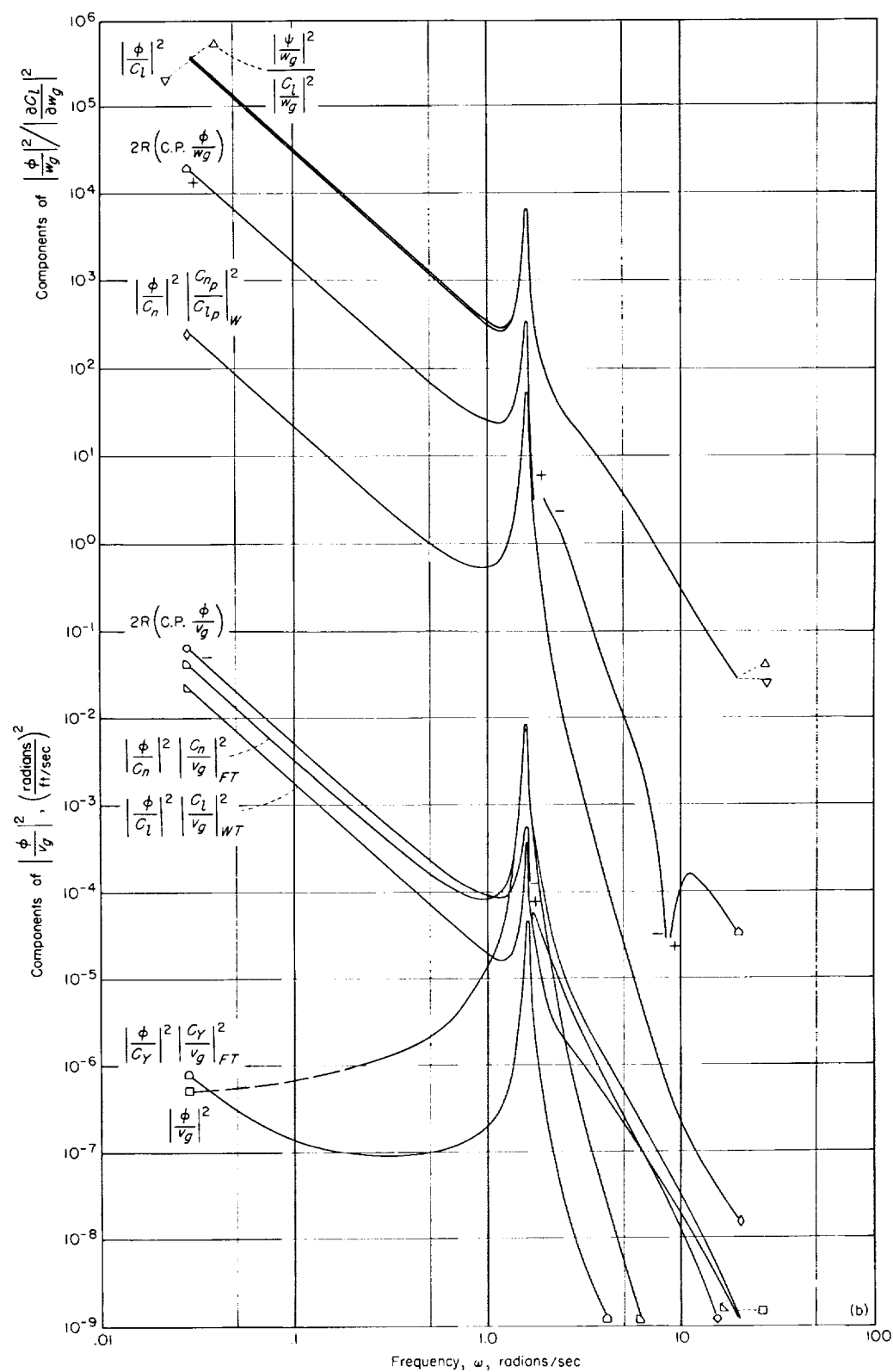
(b) Sideslip angle due to forces and moments induced by v_x and w_x components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 5. —Concluded.



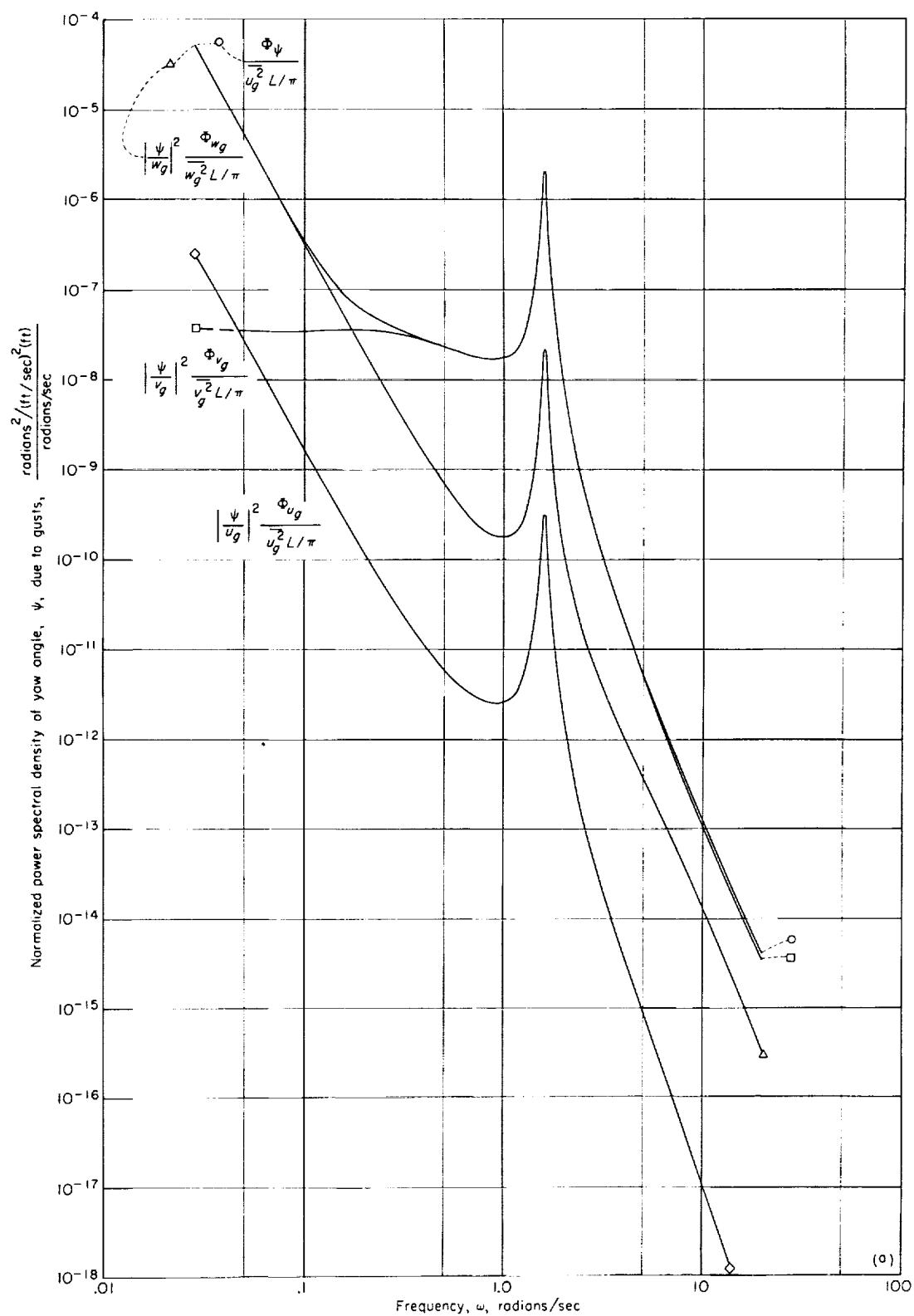
(a) Components and sum of components of power spectral density of roll angle due to three components of gust velocity.

FIGURE 6.—Response in roll angle of airplane B flying through continuous atmospheric turbulence.



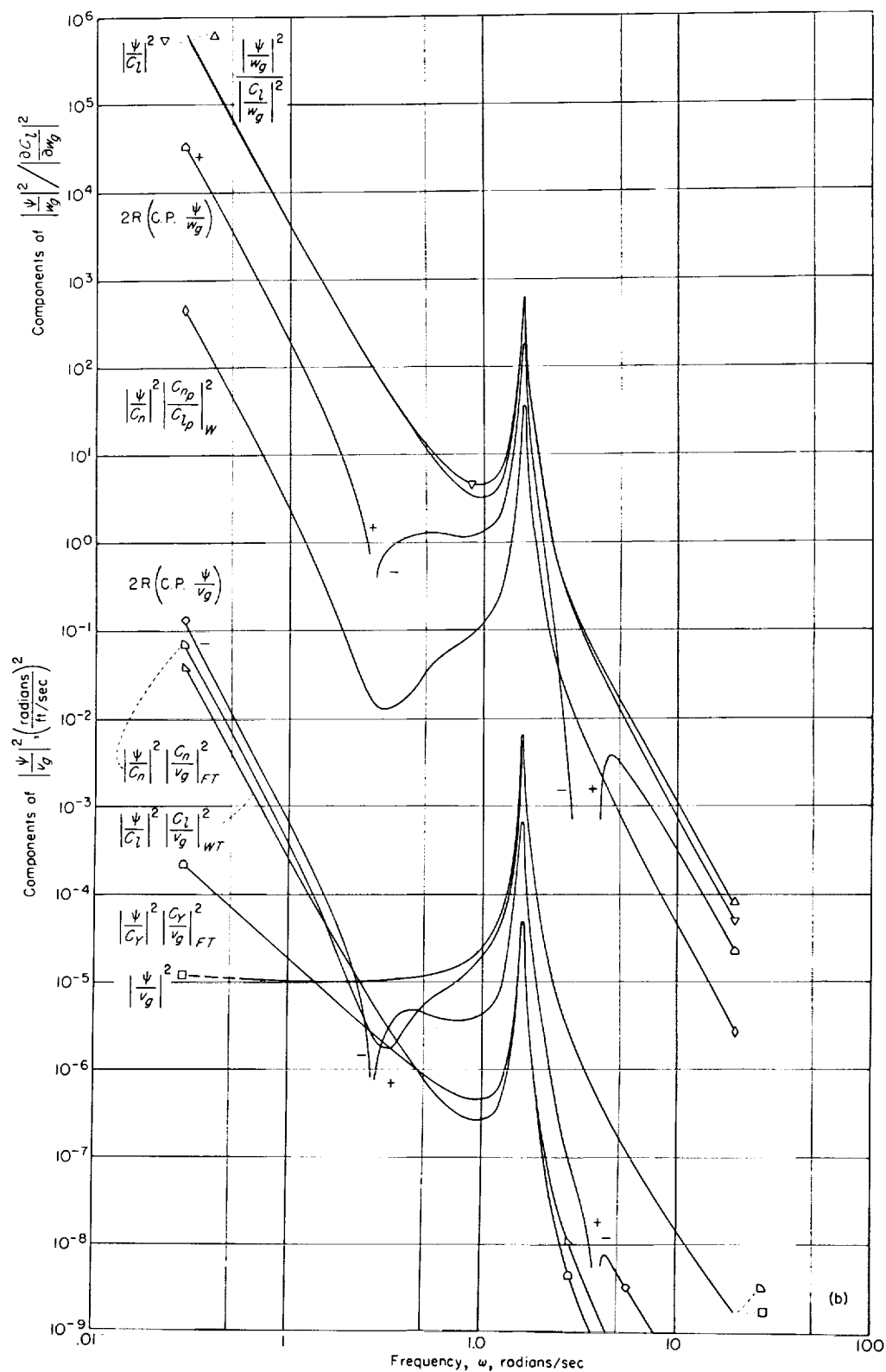
(b) Roll angle due to forces and moments induced by v_x and w_x components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 6.—Concluded.



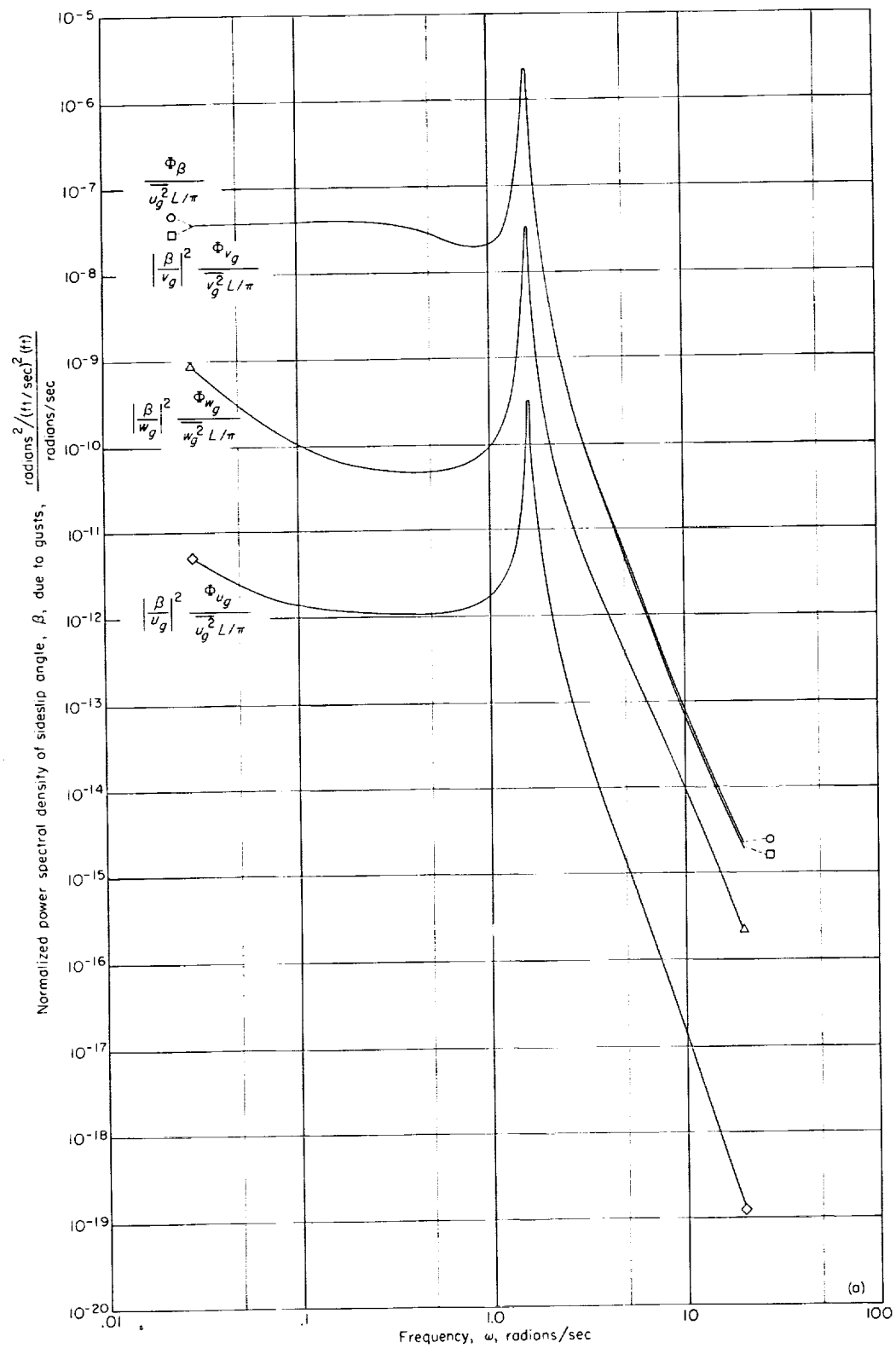
(a) Components and sum of components of power spectral density of yaw angle due to three components of gust velocity.

FIGURE 7.—Response in yaw angle of airplane B flying through continuous atmospheric turbulence.



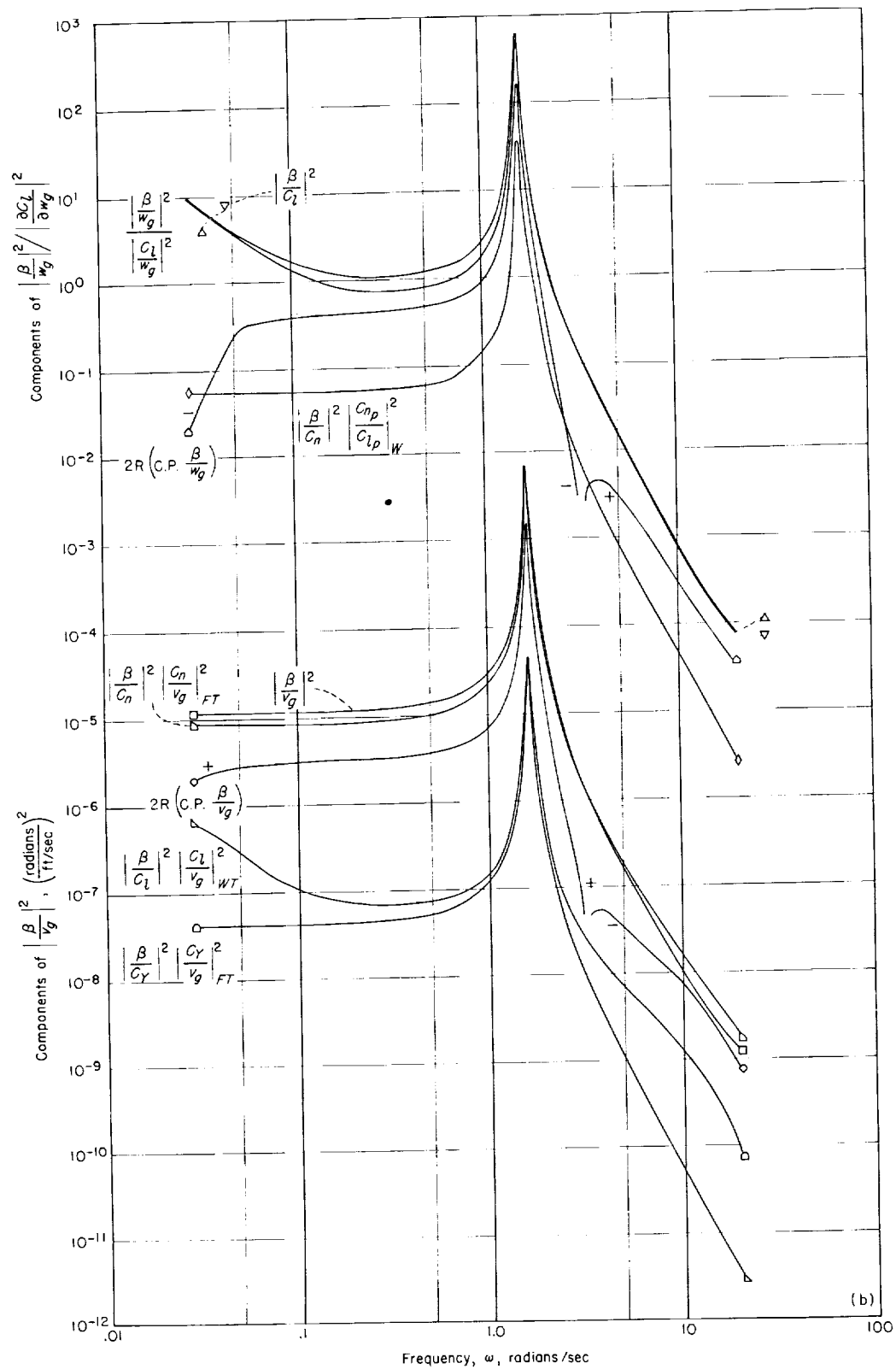
(b) Yaw angle due to forces and moments induced by v_x and w_x components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 7.—Concluded.



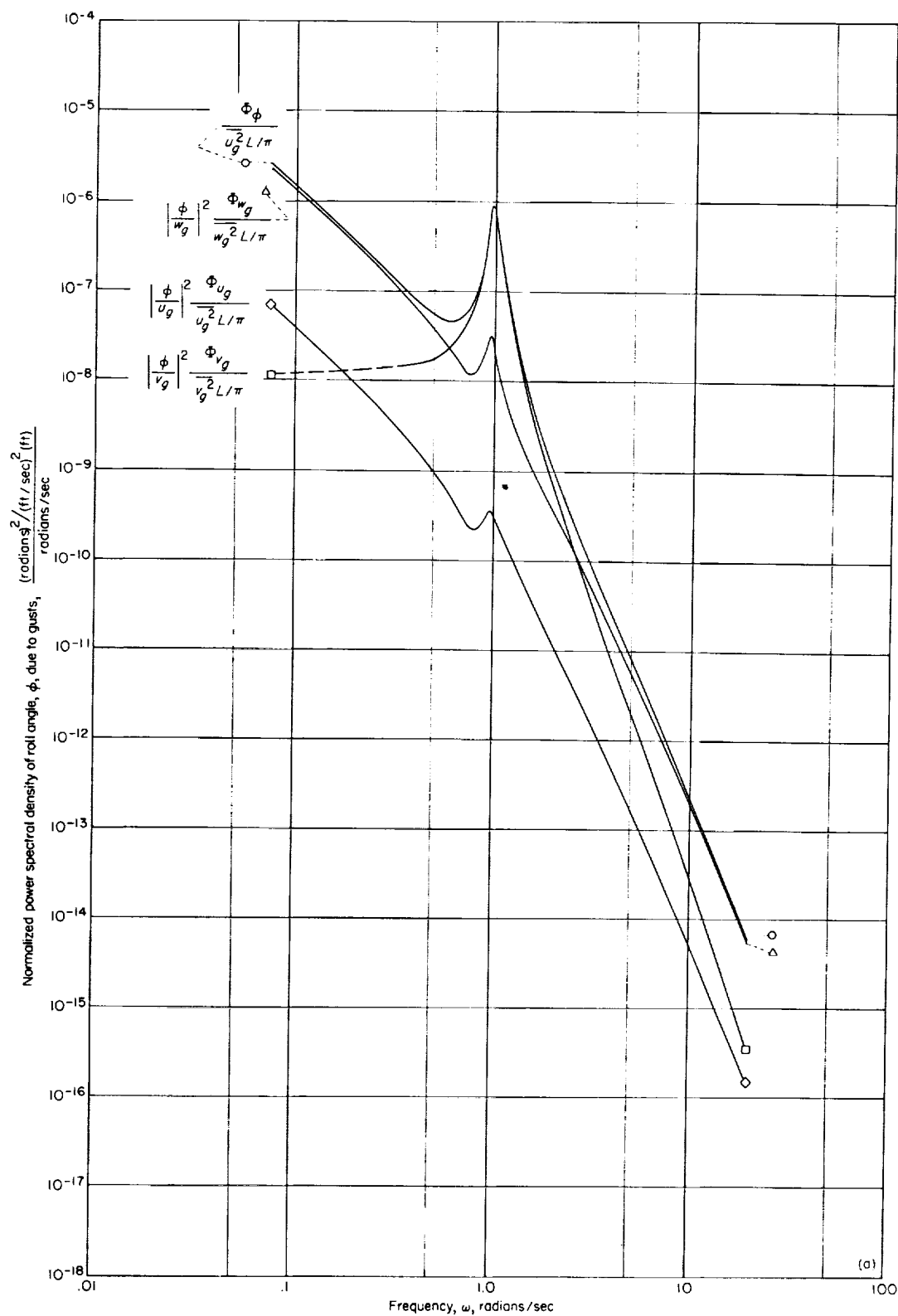
(a) Components and sum of components of power spectral density of sideslip angle due to three components of gust velocity.

FIGURE 8.-- Response in sideslip angle of airplane B flying through continuous atmospheric turbulence.



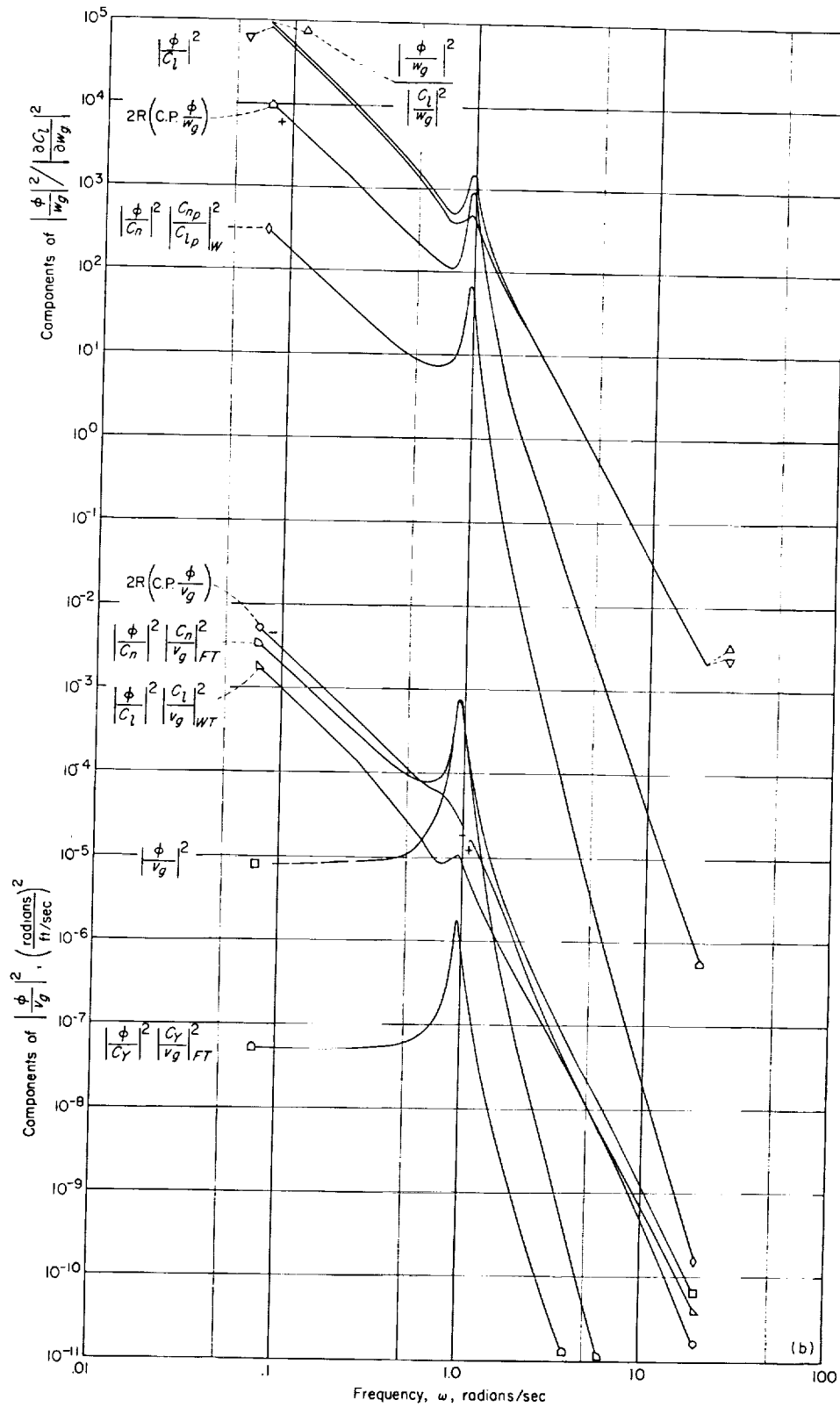
(b) Sideslip angle due to forces and moments induced by v_x and w_x components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 8. Concluded.



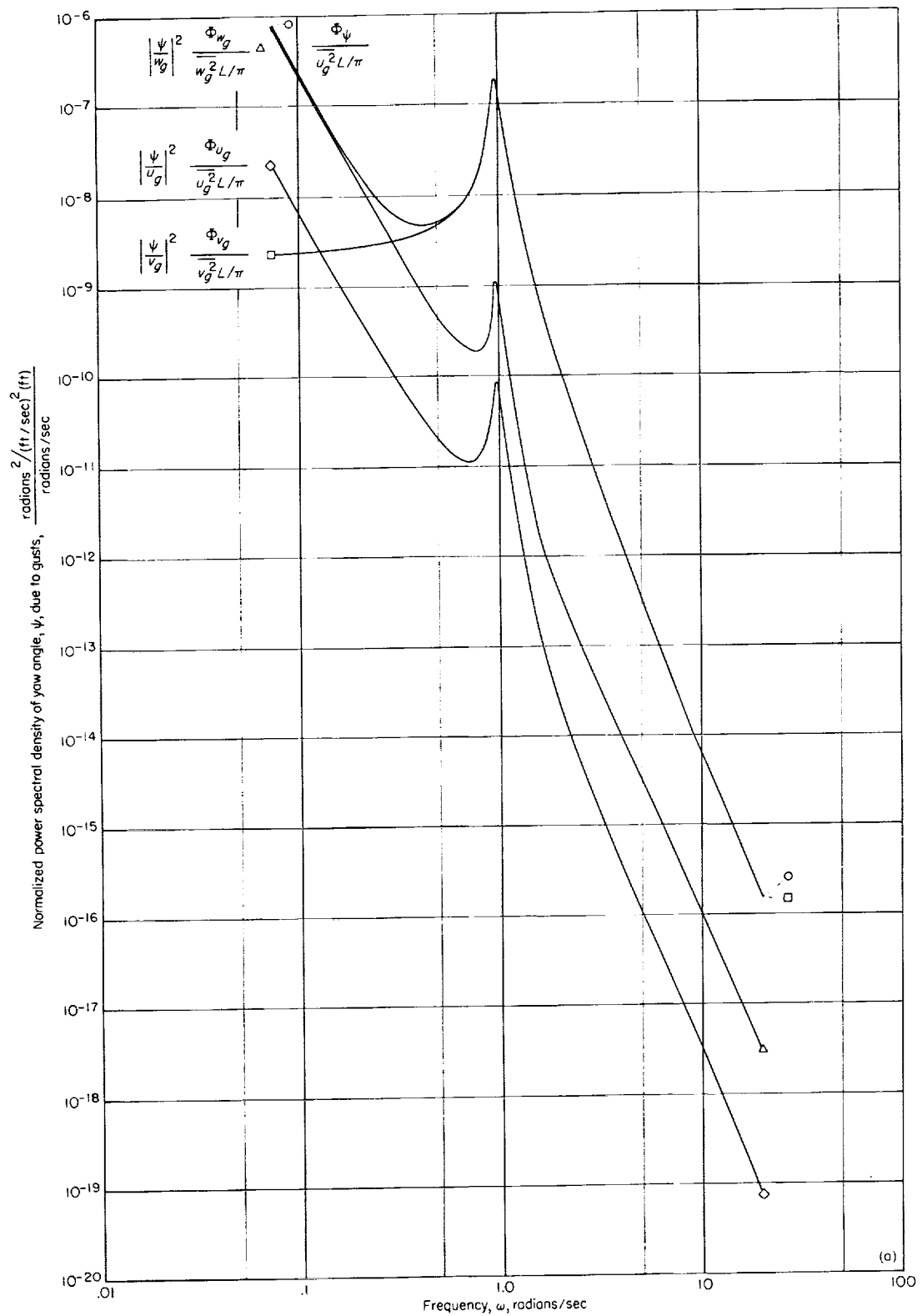
(a) Components and sum of components of power spectral density of roll angle due to three components of gust velocity.

FIGURE 9.—Response in roll angle of airplane C flying through continuous atmospheric turbulence.



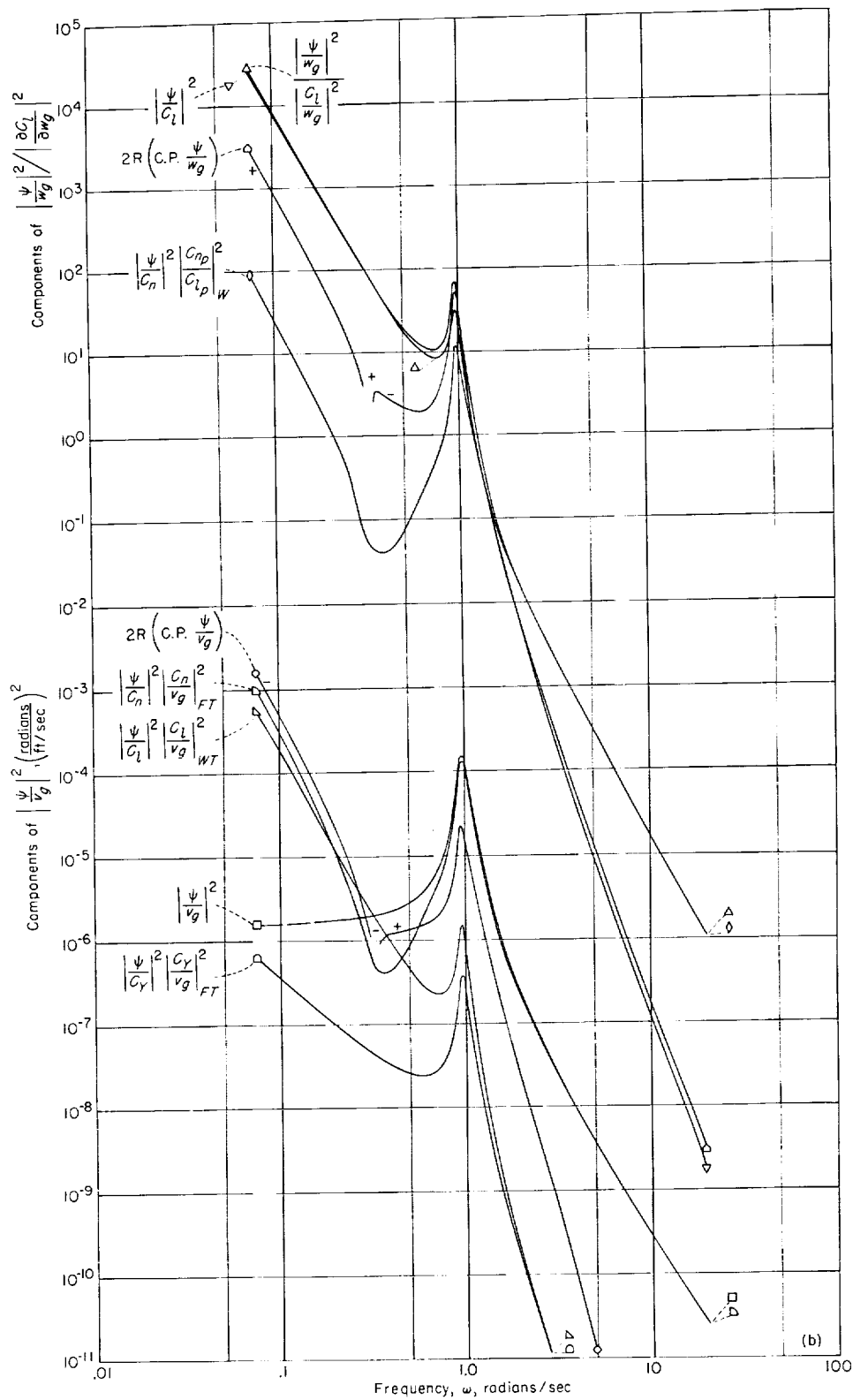
(b) Roll angle due to forces and moments induced by v_s and w_s components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 9.— Concluded.



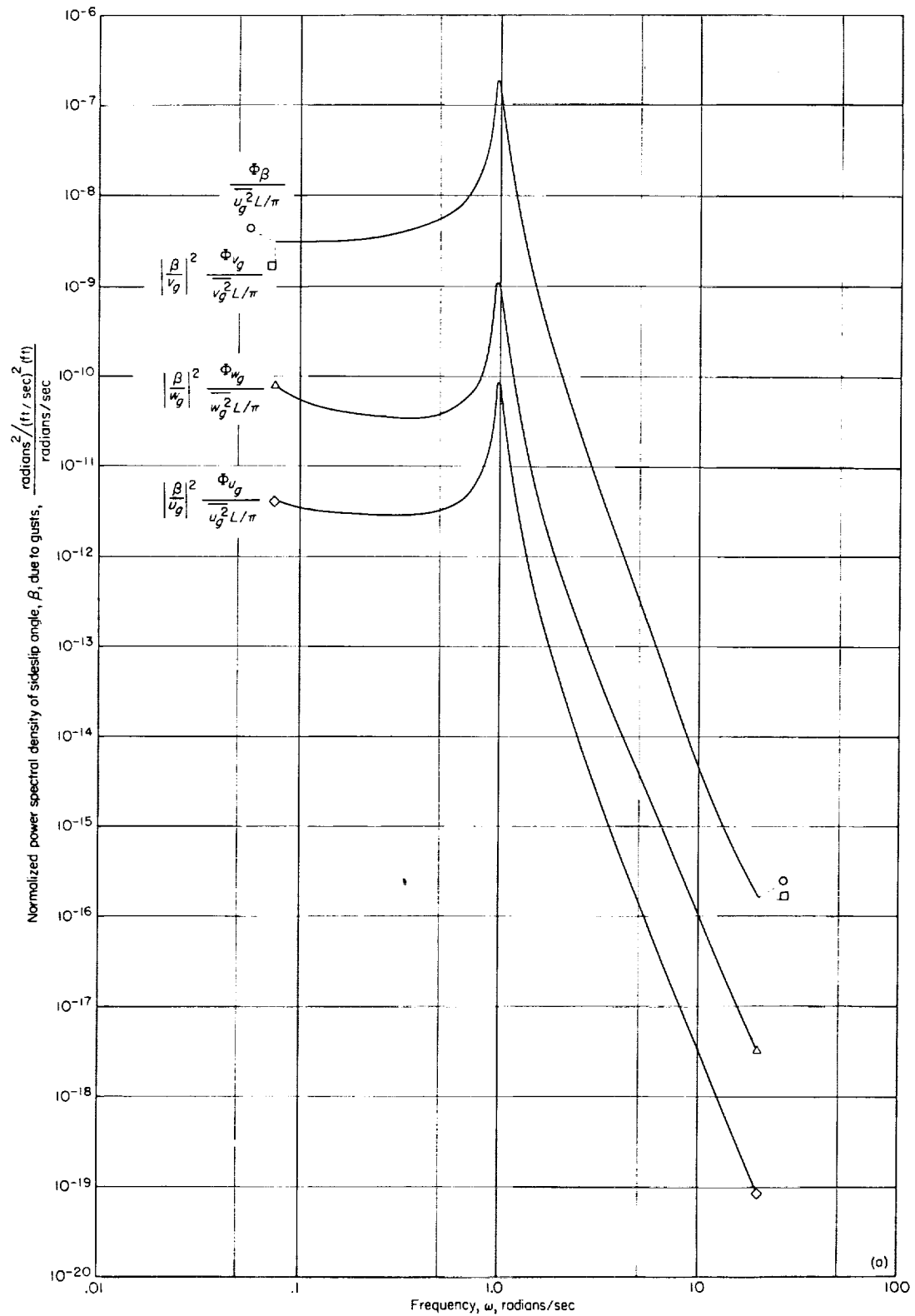
(a) Components and sum of components of power spectral density of yaw angle due to three components of gust velocity.

FIGURE 10.—Response in yaw angle of airplane C flying through continuous atmospheric turbulence.



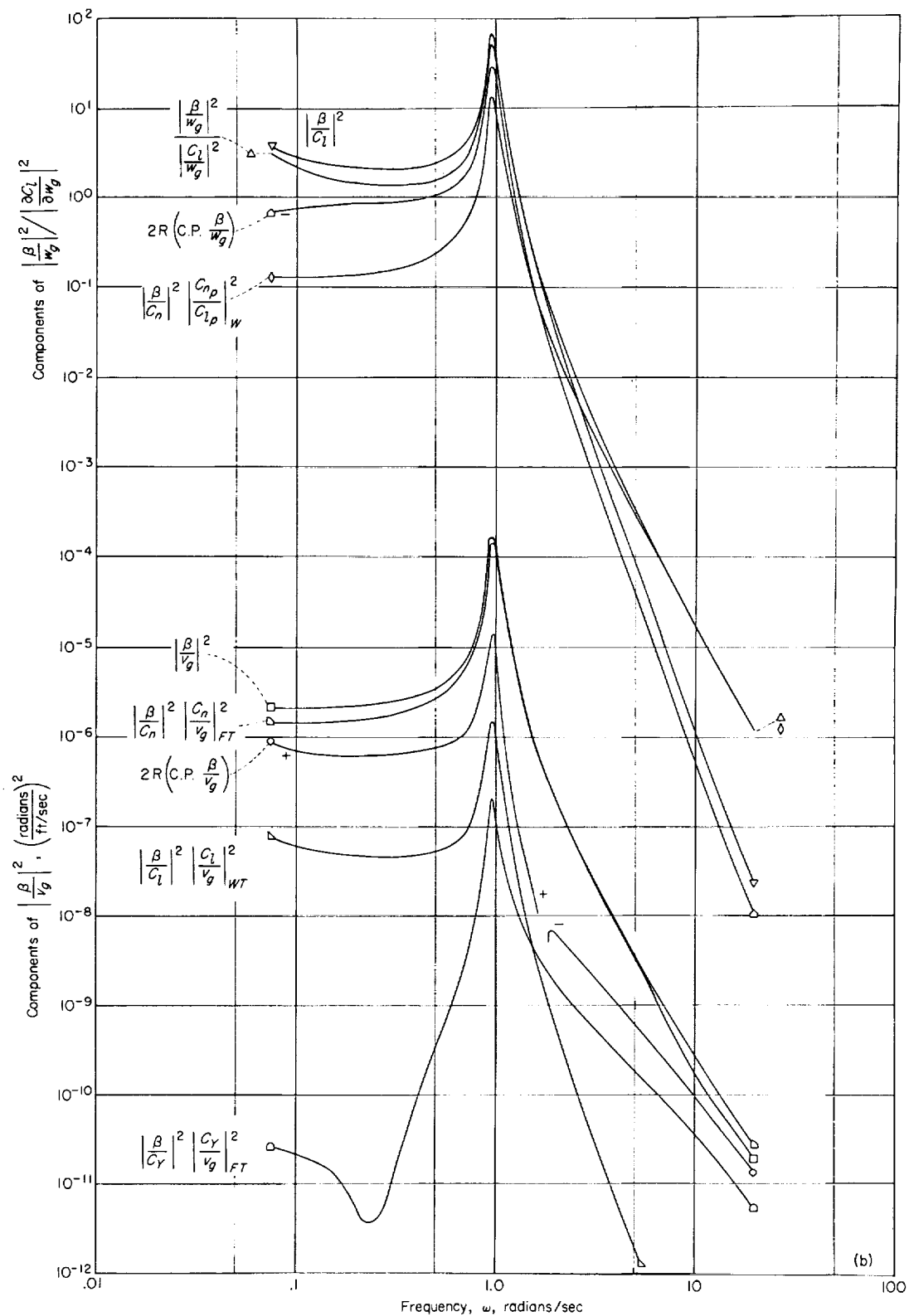
(b) Yaw angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 10.—Concluded.



(a) Components and sum of components of power spectral density of sideslip angle due to three components of gust velocity.

FIGURE 11.—Response in sideslip angle of airplane C flying through continuous atmospheric turbulence.



(b) Sideslip angle due to forces and moments induced by v_x and w_x components of gust velocity. Shown are sum and relative magnitudes of parts.

FIGURE 11.—Concluded.

A second characteristic of the method is the inaccuracy of equation (20) as $\omega \rightarrow 0$ for the ϕ and ψ modes of motion. The reason for this inaccuracy becomes apparent when the equation relating ϕ (or ψ) to v_g

$$\frac{\phi}{v_g}(\omega) = \frac{\phi}{C_l} \frac{C_l}{v_g}(\omega) + \frac{\phi}{C_n} \frac{C_n}{v_g}(\omega) + \frac{\phi}{C_Y} \frac{C_Y}{v_g}(\omega) \quad (51)$$

is written in the form

$$\begin{aligned} \frac{\phi}{v_g}(\omega) = & \frac{1}{U} \left\{ \frac{\phi}{C_l} \left[C_{l\beta} + \frac{C_l}{\beta_g}(\omega) \right] + \frac{\phi}{C_n} \left[C_{n\beta} + \frac{C_n}{\beta_g}(\omega) \right] \right. \\ & \left. + \frac{\phi}{C_Y} \left[C_{Y\beta} + \frac{C_Y}{\beta_g}(\omega) \right] \right\} \\ \frac{\phi}{v_g}(\omega) = & \frac{1}{U} \left(\frac{\phi}{C_l} C_{l\beta} + \frac{\phi}{C_n} C_{n\beta} + \frac{\phi}{C_Y} C_{Y\beta} \right) + \frac{1}{U} \left[\frac{\phi}{C_l} \frac{C_l}{\beta_g}(\omega) \right. \\ & \left. + \frac{\phi}{C_n} \frac{C_n}{\beta_g}(\omega) + \frac{\phi}{C_Y} \frac{C_Y}{\beta_g}(\omega) \right] \quad (52) \end{aligned}$$

The first half of equation (52) now represents the steady-state response to a uniform change in side gust over the airplane. The second half represents the additional response due to the nonuniform distribution of the gusts over the airplane. If the frequency-dependent expressions for the transfer functions $\frac{\phi}{C_l}$, $\frac{\phi}{C_n}$, and $\frac{\phi}{C_Y}$ as given by equations (26), (29), and (32), respectively, are substituted into equation (52) and the coefficients having the same powers of ω are grouped ($\frac{C_l}{\beta_g}(\omega)$ may be expanded in the form $\sum_{m=1}^{\infty} a_m \omega^m$), the coefficient of the lowest order power of ω in the numerator may be shown to be identically equal to zero. In the numerical evaluations at low frequencies where the lowest order terms of equation (52) (or the square of the absolute value of eq. (52)) are predominant, small differences between large numbers appear unless these terms are canceled beforehand. Since in the present method this cancellation is not convenient, a scattering of calculated values for $\frac{\phi}{v_g}$ and $\frac{\psi}{v_g}$ begins to occur at some frequency below that of the Dutch roll mode. However, since this scattering occurs only as $\frac{\phi}{v_g}$ and $\frac{\psi}{v_g}$ approach a constant (at $\omega=0$), this constant may be evaluated and the curve faired to that value.

In figures 3, 4, 6, 7, 9, and 10 this fairing is denoted with dashed lines. In the range of frequencies where the response in ϕ and ψ due to v_g is faired, the total response of the three airplanes is due almost entirely to the w_g component of the gusts.

Effect of trim angle of attack.—In analyzing the motions of the three example airplanes, the effect of small differences in trim lift coefficients or trim angles of attack appears to be important. Although the trim angles of attack for all three airplanes are low, their relative magnitudes are

$$B:A:C = 1:1.36:2.50$$

Thus, airplane C was at an angle of attack from two to two and a half times as great as airplanes A and B. The stability derivatives for the wing alone in incompressible flow are functions of angle of attack (and, hence, lift coefficient) of the order (see ref. 20):

$$C_{l_p} = \text{Constant}$$

$$C_{n_p} \propto \alpha$$

$$C_{l_r} \propto \alpha$$

$$C_{n_r} \propto \alpha^2 \text{ and } C_{D,o}$$

The effect of increased angle of attack on the ratio of yawing moment to rolling moment may be seen to increase by

$$\frac{C_{n_p}}{C_{l_p}} \propto \alpha \quad \frac{C_{n_r}}{C_{l_r}} \propto \alpha \text{ and } \frac{C_{D,o}}{\alpha}$$

These ratios may explain trends such as that shown in the case of airplane C for which the response in ψ and β to w_g at the higher frequencies is primarily due to the ability of w_g to produce yawing moment, whereas for airplanes A and B the response is primarily due to the ability of w_g to produce rolling moment.

Effect of horizontal gusts.—While there is, in general, some cross power (or cross correlation) between any two velocity components measured at any two points in the turbulence, only those components lying in the XY-plane of the airplane are considered herein since the Z-gradients of the gusts are unimportant to the airplane. The airplane will, however, sense the correlation between the u_g component acting at one point on the wing

(at any instant) and the v_g component acting at some other point on the wing or vertical tail (a vortex effect). It is the additional effect of this correlation that has been neglected in the method of this paper. While this correlation is probably small, the fact that u_g has a negligible effect on the airplanes considered is reason enough for neglecting any cross-power effects involving u_g . In cases where the u_g component of gusts does have a pronounced effect on the airplane, the contribution of the cross power must be determined or further justification be made for neglecting it.

Response characteristics of example airplanes.

The calculated lateral responses of all three example airplanes exhibited a number of characteristics in common. Most important of these was the total motion in each of the three degrees of freedom.

In each case the response in roll was due primarily to w_g at low frequencies, v_g near the Dutch roll frequency, and u_g at the higher frequencies. The response of the example airplanes in yaw was due primarily to w_g at very low frequencies and due primarily to v_g at the Dutch roll and higher frequencies. The response in β of the three example airplanes was due primarily to v_g throughout the frequency spectrum.

SIMPLIFIED EQUATIONS

Based on the three conventional example airplanes, certain simplifications to the response equations (18) to (21) appear to be justified. For the low trim lift coefficients (or angles of attack) investigated, the effect of the horizontal gust (u_g) was not discernible in any of the three modes of motion ((a) parts of figs. 3 to 11) and could have been neglected. Furthermore, in every case investigated, the effect of side force due to side gusts ($\frac{C_Y}{v_g}$) was negligible ((b) parts of figs. 3 to 11) apparently because of the magnitude of $C_{Y\beta}$ and the magnitude of the $\frac{\phi}{C_Y}$, $\frac{\psi}{C_Y}$, and $\frac{\beta}{C_Y}$ transfer functions for the example airplanes; however, these orders of magnitude are typical of most present-day airplanes. Other parameters appear to be negligible in some cases. When computing the response of the airplane in sideslip, neglecting the effect of w_g as well as that of u_g appears to

be justified. Although for some cases, for example, airplanes A and B, the effect of $\frac{C_n}{w_g}$ appears to be of small order in all modes, this effect is not generally true, as is shown by airplane C. As pointed out in a preceding section, the response in sideslip was due primarily to v_g .

In general, then, the preceding discussion leads to the simplification of equations (18) to (21) as follows, where ϕ may again be replaced by any of the lateral degrees of freedom in the form of angular displacements, velocities, or accelerations:

$$\Phi_\phi = \left| \frac{\phi}{v_g} \right|^2 \Phi_{v_g} + \left| \frac{\phi}{w_g} \right|^2 \Phi_{w_g} \quad (53)$$

and

$$\begin{aligned} \left| \frac{\phi}{v_g} \right|^2 = & \left| \frac{\phi}{C_l} \right|^2 \left| \frac{C_l}{v_g} \right|_w^2 + \left| \frac{\phi}{C_l} \right|^2 \left| \frac{C_l}{v_g} \right|_T^2 \\ & + \left| \frac{\phi}{C_n} \right|^2 \left| \frac{C_n}{v_g} \right|_{FT}^2 + 2R \left\{ \left[\left(\frac{C_l}{v_g} \right)_w^* \left(\frac{C_n}{v_g} \right)_{FT} \right. \right. \\ & \left. \left. + \left(\frac{C_l}{v_g} \right)_T^* \left(\frac{C_n}{v_g} \right)_{FT} \right] \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \right. \\ & \left. + \left(\frac{C_l}{v_g} \right)_w^* \left(\frac{C_l}{v_g} \right)_T \left| \frac{\phi}{C_l} \right|^2 \right\} \quad (54) \end{aligned}$$

$$\begin{aligned} \left| \frac{\phi}{w_g} \right|^2 = & \left\{ \left| \frac{\phi}{C_l} \right|^2 + \left| \frac{\phi}{C_n} \right|^2 \left(\frac{C_{n_p}}{C_{l_p}} \right)_w^2 \right. \\ & \left. + 2R \left[\left(\frac{C_{n_p}}{C_{l_p}} \right)_w \left(\frac{\phi}{C_l} \right)^* \frac{\phi}{C_n} \right] \right\} \left| \frac{C_l}{w_g} \right|_w^2 \quad (55) \end{aligned}$$

COMPARISON OF METHODS

Both airplanes A and B have been analyzed in other papers (refs. 6 to 9) at the same flight condition as that used herein for their responses to atmospheric turbulence by using methods which treat gusts as equivalent motions of the airplane in still air. These gusts are denoted as side gusts β_g , rolling gusts $D\phi_g$ or $\frac{\partial w_g}{\partial y}$, and yawing gusts $D\psi_g$ or $\frac{\partial u_g}{\partial y}$. The analysis of airplane A given in reference 6 is based on the power spectra derived from the analysis of reference 9. Although the analysis of reference 6 also incorporates a refinement by considering the lag of the vertical tail penetrating the gust, the effect proved to be negligible on such a small fast airplane. Since in this analysis (ref. 6) the rolling gust affected only

the wing, a comparison with the analysis of this paper (where the vertical gust likewise affects only the wing) may be made. Therefore, a comparison is made herein by using airplane A to illustrate any differences in the results of the two methods.

COMPARISON OF GUST POWER SPECTRA

Since the power spectrum of the v_g component of atmospheric turbulence as used in references 9 and 6 and that used herein have in common an asymptotic value of k^{-2} at large values (relatively speaking) of k , the arbitrary constant A , given in references 9 and 6, is assigned a value such that the referenced v_g spectrum and the v_g spectrum of this report are asymptotically congruent. With this same constant, the $D\phi_g$ and $\frac{\partial w_g}{\partial y}$ spectra are compared, and all comparisons of the calculated motions of airplane A are likewise based on this same constant.

Side gusts.—The power spectrum of side gusts expressed by equation (47) written in the form

$$\frac{\Phi_{v_g}}{v_g^2} = \frac{\Phi_{\beta_g}}{v_g^2/U^2} = \frac{L}{\pi U} \frac{1 + 3\left(\frac{\omega L}{U}\right)^2}{\left[1 + \left(\frac{\omega L}{U}\right)^2\right]^2} \quad (56)$$

is comparable to that shown in figure 4 of reference 6 except for a constant. That is,

$$\frac{\Phi_{v_g}}{v_g^2} \approx \frac{\Phi_{\beta_g}}{A} \quad (57)$$

where

$$A = K \frac{\overline{v_g^2}}{U^2} \quad (58)$$

Plotted in figure 12 are the side-gust spectrum

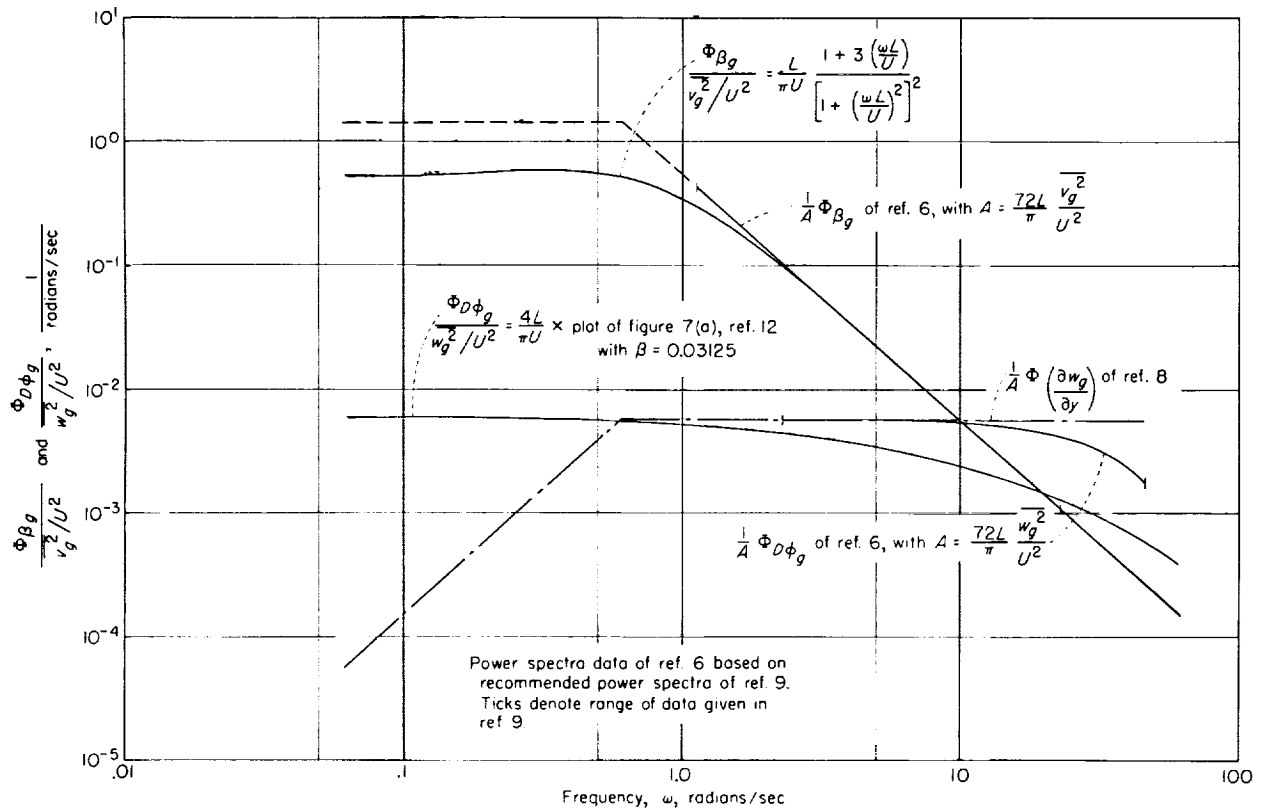


FIGURE 12.—Comparison of several theories for power spectra of side gusts and rolling gusts as experienced by airplane A. $U = 696$ ft/sec; $b = 35.25$ feet; $L = 1,128$ feet.

given by equation (56) and that of reference 6 with the constant K of equation (58) given the value

$$K = \frac{72L}{\pi} \quad (59)$$

in order to get the agreement shown. Since both spectra are multiplied by the factor L , this agreement is independent of the choice of magnitude for L .

Rolling gusts.—Similarly, the power spectrum of rolling gusts $D\phi_g$ used in reference 6 may be compared with the power spectrum of rolling moment due to vertical gusts as given in reference 12 and used herein. Since

$$C_l = -C_{l_p} \frac{pb}{2U} = -\frac{C_{l_p}}{2} D\phi \quad (60)$$

then

$$\Phi_{D\phi_g} = \frac{4}{C_{l_p}^2} \Phi_{C_l} \quad (61)$$

and for airplane A, under the conditions considered herein,

$$\Phi_{D\phi_g} = \frac{4}{C_{l_p}^2} \frac{\overline{w_g^2} L C_{l_p}^2}{\pi U^3} E(w_g)$$

where $E(w_g)$ is the curve of $\frac{\Phi_{C_l}(k')}{\overline{w_g^2} L C_{l_p}^2 / U^3 \pi}$ for $\beta' = 0.03125$ from reference 12, figure 7(a). If $\Phi_{D\phi_g}$ is expressed with the same normalization used for the side gust (see eq. (56)), then

$$\frac{\Phi_{D\phi_g}}{\overline{w_g^2} U^2} = \frac{4L}{\pi U} E(w_g) \quad (62)$$

Equation (62) is plotted in figure 12 together with the power spectrum of $D\phi_g$ given in reference 6 again with

$$A = \frac{72L}{\pi} \frac{\overline{v_g^2}}{U^2} \quad (63)$$

Since the spectra of reference 6 are the same as those given in reference 9 except for a slight difference in frequency range, no difference is shown in the plots except that the range of data of reference 9 is denoted with ticks.

Likewise shown in figure 12 is the spectrum for rolling gusts as predicted by Decaulne in reference 8. The theory is based on the assumption that

the variation of vertical gusts across the span $(\partial w_g / \partial y)$ can be approximated with a constant gradient (a linear variation) and has as a spectrum

$$\Phi_{\frac{\partial w_g}{\partial y}}(\omega) = \Phi_{w_g}(\omega) \frac{36}{b^2} \left[\frac{\sin \frac{\omega b}{2U}}{\left(\frac{\omega b}{2U}\right)^2} - \frac{\cos \frac{\omega b}{2U}}{\frac{\omega b}{2U}} \right]^2 \quad (64)$$

which for small values of ω reduces to

$$\Phi_{\frac{\partial w_g}{\partial y}}(\omega) = \Phi_{w_g}(\omega) \left(\frac{\omega}{U} \right)^2 \quad (65)$$

Based on this theory, when ω is less than the frequency corresponding to a wavelength of L (a value of L of 1,128 feet used in fig. 12), the vertical-gust spectrum will become a constant and the $\partial w_g / \partial y$ spectrum will fall off to zero at the rate of ω^2 as ω approaches zero. This result is denoted by dashed lines in figure 12.

At the higher frequencies shown in figure 12, the calculated spectrum of reference 8 predicts almost no attenuation of rolling power of the wing with frequency. The spectrum of the same quantity according to references 9 and 6 would predict an attenuation for the wing of airplane A, the attenuation beginning at about 8 radians/sec with the power falling off rapidly thereafter. The more comprehensive theory of reference 12 shows a more pronounced attenuation of power over the plotted frequency range although all theories have approximately the same value at the frequency corresponding to the value of L .

On the basis of this comparison, the concept of constant gust gradients across the span would appear to be insufficient, despite the agreement in the region of the break frequency $\left(\omega = \frac{U}{L}\right)$. The more comprehensive theory of random gusts across the span predicts no attenuation below the break frequency such as is predicted by the theories of references 9 and 6. At frequencies greater than the break frequency, the attenuation predicted by the two theories again becomes inconsistent.

COMPARISON OF LATERAL RESPONSE POWER SPECTRA

With the assigned value for the constant A given by equation (63), the responses of airplane A in ϕ , ψ , and β due to β_g and $D\phi_g$ as calculated in reference 6 (fig. 10(a)) are plotted in figures 13 (a), (b), and (c), respectively. Also replotted in

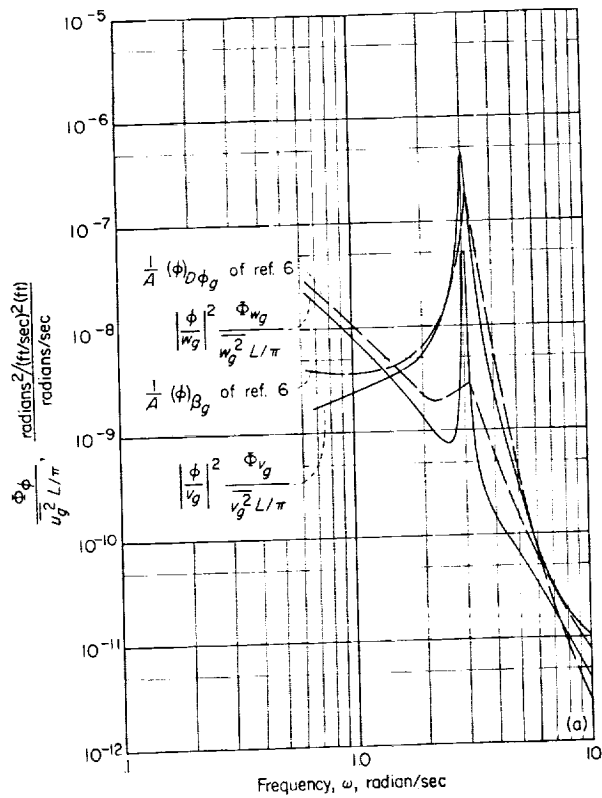
(a) Power spectral response in roll angle ϕ of airplane A.

FIGURE 13. Comparison between the lateral responses of airplane A in atmospheric turbulence as determined by method of present paper and method of reference 6.

figure 13 are the responses in ϕ , ψ , and β of airplane A due to v_g and w_g as obtained by the more comprehensive method of this paper. Since reference 6 did not consider the u_g component of gust velocity, no comparison is made of the contribution of this component. Flight conditions are the same in both cases.

In reproducing the frequency-response calculations of reference 6, the calculated transfer functions of $\psi/D\phi_g$ were found to be in error; however, this error did not affect the conclusions made therein. The power spectral response of $(\psi)_{D\phi_g}$ shown in figure 13(b) has therefore been corrected and plotted for a somewhat larger frequency range than is given in reference 6. In the extended frequency range the $D\phi_g$ spectrum which would be predicted by equation (65) is used.

At the lower frequencies the differences in the response of the airplane due to side gusts as calculated by the two methods are due principally to the differences in the spectra of side gusts used

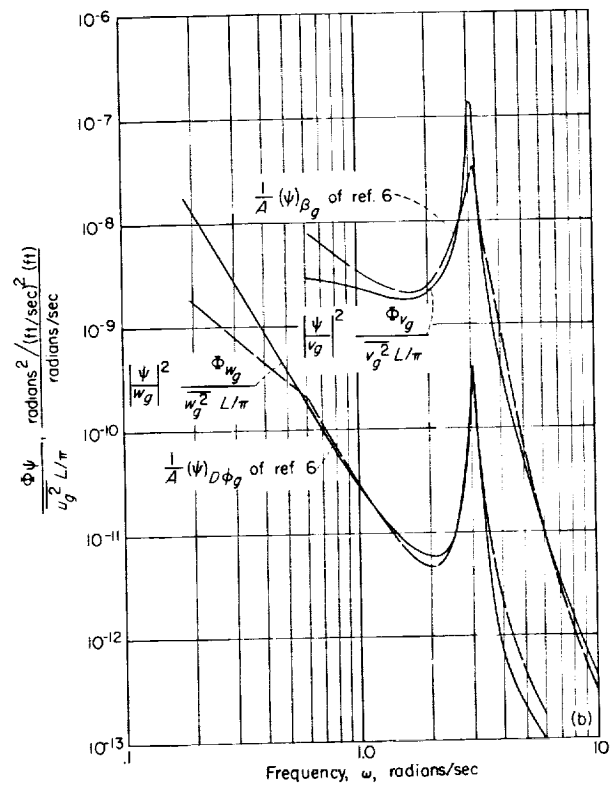
(b) Power spectral response in yaw angle ψ of airplane A.

FIGURE 13. Continued.

by the two methods, as shown in figure 12. In the intermediate frequency range where the Dutch roll mode is predominant, a large difference in peak amplitudes is indicated by the two methods. This difference appears to be due to failure to evaluate the transfer functions at the exact frequency at which the Dutch roll mode has maximum frequency content ($\omega_n = 3.16$ radians/sec for ref. 6 as compared with $\omega_n = 3.08$ radians/sec for this paper). In power spectral analyses where frequency amplitudes are squared, this factor can result in appreciable differences, as demonstrated in figure 13.

In the higher frequency range the differences in the theories are more pronounced since in this range the calculated effectiveness of the vertical gusts acting on the wing to produce lateral moments differs with the two theories. As pointed out in reference 11, the distribution of gust velocities along the fuselage begins to become an important factor in calculating the yawing moment and side force due to side gusts in this frequency range.

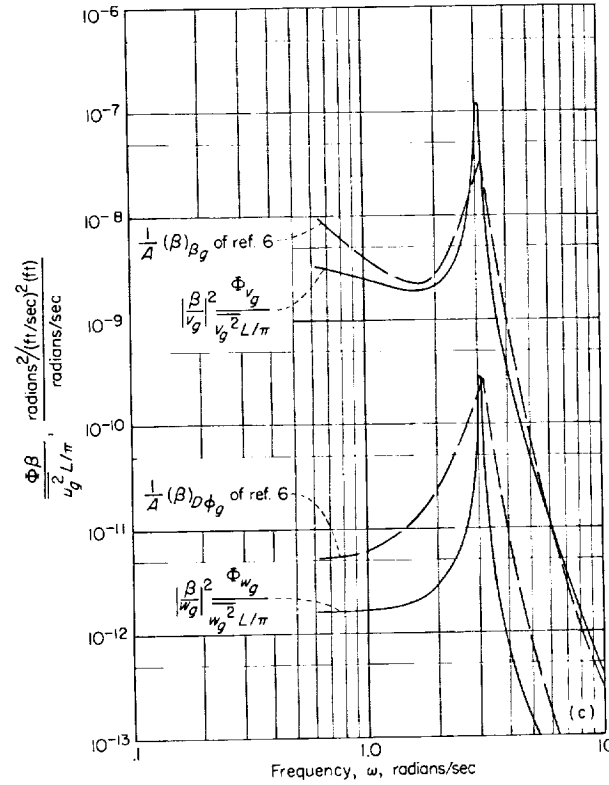
(c) Power spectral response in sideslip angle β of airplane A.

FIGURE 13. Concluded.

II. A SIMPLIFIED METHOD FOR THE CALCULATION OF THE LATERAL RESPONSE OF AIRPLANES TO RANDOM TURBULENCE

THEORY

EQUATIONS OF MOTION

The equations of lateral motion for an airplane (ref. 22) are given by

$$\left. \begin{aligned} 2\mu K_X^2 D^2 \phi_o - \frac{1}{2} C_{i_p} D \phi - 2\mu K_{XZ} D^2 \psi_o - \frac{1}{2} C_{i_r} D \psi - C_{i_\beta} \beta &= 0 \\ -2\mu K_{XZ} D^2 \phi_o - \frac{1}{2} C_{n_p} D \phi + 2\mu K_Z^2 D^2 \psi_o - \frac{1}{2} C_{n_r} D \psi - C_{n_\beta} \beta &= 0 \\ -\frac{1}{2} C_{Y_p} D \phi - C_L \phi_o + 2\mu D \psi_o - \frac{1}{2} C_{Y_r} D \psi - C_L \tan \gamma \psi_o + 2\mu D \beta_o - C_{Y_\beta} \beta &= 0 \end{aligned} \right\} \quad (66)$$

where the subscript o appearing in the inertial and weight terms is used to denote angular displacement with respect to an absolute system of axes fixed in the general air mass. In calculations of the motion of an airplane in still air, the angular displacements and velocities appearing in the aerodynamic terms are identical with these values. When flying in turbulent air, however, the airplane is subjected to the motion of local air masses,

generally referred to as gusts. The relative linear and angular velocities of the airplane with respect to the local air mass each may be considered as made up of two parts:

$$\left. \begin{aligned} D\phi &= D\phi_o + D\phi_g \\ D\psi &= D\psi_o + D\psi_g \\ \beta &= \beta_o + \beta_g \end{aligned} \right\} \quad (67)$$

where the subscript g is used to denote gust velocities. Substituting equations (67) into equations (66) and transposing the terms resulting from gust disturbances to the right-hand side of the equation gives the result written in the convenient matrix form:

$$[\Delta] \begin{Bmatrix} \phi_o \\ \psi_o \\ \beta_o \end{Bmatrix} = [G] \begin{Bmatrix} D\phi_g \\ D\psi_g \\ \beta_g \end{Bmatrix} \quad (68)$$

The matrix

$$[\Delta] = \begin{bmatrix} 2\mu K_x^2 D^2 - \frac{1}{2} C_{l_p} D & -2\mu K_{xz} D^2 - \frac{1}{2} C_{l_r} D & -C_{l_\beta} \\ -2\mu K_{xz} D^2 - \frac{1}{2} C_{n_p} D & 2\mu K_z^2 D^2 - \frac{1}{2} C_{n_r} D & -C_{n_\beta} \\ -\frac{1}{2} C_{Y_p} D - C_L & \left(2\mu - \frac{1}{2} C_{Y_r}\right) D - C_L \tan \gamma & 2\mu D - C_{Y_\beta} \end{bmatrix} \quad (69)$$

is the familiar "still air" rigid-airframe characteristic equation, and the matrix

$$[G] = \begin{bmatrix} \frac{1}{2} C_{l_p} & \frac{1}{2} C_{l_r} & C_{l_\beta} \\ \frac{1}{2} C_{n_p} & \frac{1}{2} C_{n_r} & C_{n_\beta} \\ \frac{1}{2} C_{Y_p} & \frac{1}{2} C_{Y_r} & C_{Y_\beta} \end{bmatrix} \quad (70)$$

gives the relationship between the aerodynamic moments and forces resulting from gust velocities encountered.

In the classical method of treating simple isolated gust inputs (for example, ref. 2), the elements of the G matrix would be the "still air" stability derivatives, the values in the gust-velocity matrix ($D\phi_g$, $D\psi_g$, and β_g) would be those appropriate to step or ramp functions, and the solution would be defined by

$$\begin{Bmatrix} \phi_o(\omega) \\ \psi_o(\omega) \\ \beta_o(\omega) \end{Bmatrix} = [\Delta(\omega)]^{-1} [G] \begin{Bmatrix} D\phi_g(\omega) \\ D\psi_g(\omega) \\ \beta_g(\omega) \end{Bmatrix} \quad (71)$$

where $[\Delta]^{-1}$ is the inverse of $[\Delta]$ with $D = i\bar{U}\omega$.

Unique time responses could then be obtained by taking the inverse Fourier transform of each solution.

However, when the responses of the airplane to continuous random gusts are to be considered, the gust velocities can be defined only in a statistical (power-spectral) manner and the resulting moments and forces will in turn be related only in a statistical sense. This means that the elements of matrix G cannot be evaluated in the usual sense and the effect of random distributions of gust velocities along the fuselage and across the wing span must be taken into account.

FORCES AND MOMENTS DUE TO TURBULENCE

In the application of the method to random turbulence, the sources of the forces and the moments on the airplane must be considered. As shown in reference 12, yawing and rolling moments on the wing result from gradients of the horizontal and vertical gusts. Wing rolling moments and moments and forces on the fuselage and vertical tail are produced by side gusts. Although the wing moments cannot be determined at any instant as a function of the gust velocities measured at the center of gravity, the power spectra of these moments have been determined in reference 12 as a function of the power spectrum of the vertical gust velocity as measured at one point. In isotropic turbulence, the spectrum of the side gusts measured at a point is identical to that of the vertical gusts. For this reason, the spectrum of the yawing and rolling moments may be related equally well to that of the side gusts. This procedure is used in the present analysis.

Spanwise gradient of vertical gust.—In general, the rolling moment acting at any instant results from a random distribution of vertical-gust velocity across the span which has some average linear spanwise gradient. An effective gradient which produces the rolling moment due to this random spanwise distribution may be defined by the relation

$$D\phi_g = \frac{2}{(C_{lp})_w} C_l(w_g)$$

which is derived from the equation

$$C_l(w_g) = \frac{1}{2} (C_{lp})_w D\phi_g \quad (72)$$

where $C_l(w_g)$ is the rolling moment due to vertical gusts, $(C_{lp})_w$ is the wing damping-in-roll stability derivative, and $D\phi_g$ is the equivalent rolling-gust gradient. Likewise, the vertical-gust distribution at any given instant also produces a yawing moment. In reference 12, this moment was assumed to be in phase with the rolling moment and was determined from the formula

$$C_n(w_g) = \left(\frac{C_{np}}{C_{lp}} \right)_w C_l(w_g) \quad (73)$$

Substituting equation (72) into equation (73) gives the yawing moment in terms of the effective gust gradient:

$$C_n(w_g) = \frac{1}{2} (C_{np})_w D\phi_g \quad (74)$$

Spanwise gradient of horizontal gust.—Similar arguments may be used to show that a random distribution of horizontal gusts across the span of the wing produces both rolling and yawing moments which are assumed to be in phase and related by

$$C_n(u_g) = \left(\frac{C_{nr}}{C_{lr}} \right)_w C_l(u_g) \quad (75)$$

In terms of the equivalent yawing-gust gradient $D\psi_g$, these moments are given by

$$C_l(u_g) = \frac{1}{2} (C_{lr})_w D\psi_g \quad (76)$$

$$C_n(u_g) = \frac{1}{2} (C_{nr})_w D\psi_g \quad (77)$$

These formulas (eqs. (72) to (77)) are in a convenient form for application to the present analysis in which the gusts are considered as equivalent rigid-body motions of the airplane. Moreover, since only the wing moments are influenced by the horizontal- and vertical-gust distributions across the span of the wing, these formulas completely account for the effect of these gusts on the lateral motion.

Side gusts.—The only remaining source of gust disturbance is the side gust. In this part of the report this disturbance is expressed as an equivalent sideslip $\beta_g = v_g/U$, but the method of calculating its effect is essentially the same as that used in part I. Effects due to the difference in time at which a given gust encounters different sections of the airplane, known as gust-penetration effects, are accounted for in part I by considering the relations between the aerodynamic force and moment coefficients and the gust inputs measured at the center of gravity as frequency-dependent transfer functions. These relations are herein converted to frequency-dependent stability derivatives $C_{Y\beta}$, $C_{n\beta}$, and $C_{l\beta}$ by multiplying the expressions used in part I by the flight velocity U . Expressions for these stability derivatives are given in appendix C.

To an observer in the airplane the side-gust disturbance appears as a random velocity distribution which moves along the fuselage and vertical tail at the mean velocity U . If this speed in terms of body lengths per unit time is large (as is usually the case with most airplanes), these gust velocities will not change appreciably during their "time of exposure" to the fuselage. When the condition is satisfied, the forces and moments acting on the fuselage are uniquely, rather than statistically, related to the side-gust distribution along the flight path, and phase relations can be correctly accounted for by treating the frequency-dependent derivatives as complex quantities.

MATRIX SOLUTION TO EQUATIONS OF MOTION

The effects of the gusts on the wing and the fuselage-tail combination may be incorporated into a matrix similar to the simplified G matrix of equation (70). If this matrix is denoted by $\tilde{G}(\omega)$ and if only those force and moment coefficients are included which will have a significant effect (for example, side forces due to rolling and yawing are usually negligible compared with side force

due to sideslip), the modified G matrix is defined as

$$[\tilde{G}(\omega)] = \begin{bmatrix} \frac{1}{2}(C_{l_p})_w & \frac{1}{2}(C_{l_r})_w & [C_{l_\beta}(\omega)]_{wT} \\ \frac{1}{2}(C_{n_p})_w & \frac{1}{2}(C_{n_r})_w & [C_{n_\beta}(\omega)]_{rT} \\ 0 & 0 & [C_{Y_\beta}(\omega)]_{rT} \end{bmatrix} \quad (78)$$

In terms of this matrix, the solution to the equations of motion given by equation (68) is obtained formally by inversion:

$$\begin{Bmatrix} \phi_u(\omega) \\ \psi_u(\omega) \\ \beta_o(\omega) \end{Bmatrix} = [\Delta(\omega)]^{-1} [\tilde{G}(\omega)] \begin{Bmatrix} D\phi_g(\omega) \\ D\psi_g(\omega) \\ \beta_g(\omega) \end{Bmatrix} \quad (79)$$

The inverse of the Δ matrix given by equation (69) is shown in part I (eqs. (26) to (35)) to consist of transfer functions relating the response in lateral angular displacements to a sinusoidal rolling moment, yawing moment, or side force of unit amplitude:

$$[\Delta(\omega)]^{-1} = \begin{bmatrix} \frac{\phi}{C_l}(\omega) & \frac{\phi}{C_n}(\omega) & \frac{\phi}{C_Y}(\omega) \\ \frac{\psi}{C_l}(\omega) & \frac{\psi}{C_n}(\omega) & \frac{\psi}{C_Y}(\omega) \\ \frac{\beta}{C_l}(\omega) & \frac{\beta}{C_n}(\omega) & \frac{\beta}{C_Y}(\omega) \end{bmatrix} \quad (80)$$

Inasmuch as the random nature of the moments resulting from random turbulence makes it necessary to place the result given by equation (79) in a power-spectral form, both sides of the equation must be squared. If equation (79) is squared in its present form, cross-power terms between all the forces and moments will appear in the final result. However, if the product of $[\Delta]^{-1}$ and $[\tilde{G}(\omega)]$ is taken beforehand, then when the product matrix is squared these cross-power relationships do not appear explicitly since their equivalent effect has been taken into account in the multiplication process. It is shown in appendix D that the result obtained by first multiplying these two matrices before squaring is the same as the result obtained by squaring each matrix separately and including cross-power terms.

Each element in the matrix product of $[\Delta]^{-1}$

and $[\tilde{G}(\omega)]$ has the form of a transfer function relating one of the airplane response quantities to one of the gust components. This relationship is indicated as follows:

$$[\Delta^{-1}\tilde{G}(\omega)] = \begin{bmatrix} \frac{\phi}{D\phi_g}(\omega) & \frac{\phi}{D\psi_g}(\omega) & \frac{\phi}{\beta_g}(\omega) \\ \frac{\psi}{D\phi_g}(\omega) & \frac{\psi}{D\psi_g}(\omega) & \frac{\psi}{\beta_g}(\omega) \\ \frac{\beta}{D\phi_g}(\omega) & \frac{\beta}{D\psi_g}(\omega) & \frac{\beta}{\beta_g}(\omega) \end{bmatrix} \quad (81)$$

In terms of this matrix product, the relations between the power spectra of the airplane response and the gust components are given in the form

$$\begin{Bmatrix} \Phi_\phi \\ \Phi_\psi \\ \Phi_\beta \end{Bmatrix} = \begin{bmatrix} \left| \frac{\phi}{D\phi_g} \right|^2 & \left| \frac{\phi}{D\psi_g} \right|^2 & \left| \frac{\phi}{\beta_g} \right|^2 \\ \left| \frac{\psi}{D\phi_g} \right|^2 & \left| \frac{\psi}{D\psi_g} \right|^2 & \left| \frac{\psi}{\beta_g} \right|^2 \\ \left| \frac{\beta}{D\phi_g} \right|^2 & \left| \frac{\beta}{D\psi_g} \right|^2 & \left| \frac{\beta}{\beta_g} \right|^2 \end{bmatrix} \begin{Bmatrix} \Phi_{D\phi_g} \\ \Phi_{D\psi_g} \\ \Phi_{\beta_g} \end{Bmatrix} \quad (82)$$

where the power spectrum of a quantity X is defined by

$$\Phi_X = \lim_{T \rightarrow \infty} \frac{1}{T} |X|^2 \quad (83)$$

The assumption has been made that any cross power between the three gust components is either zero or negligible. Justification for this assumption is made in a subsequent section.

Finally, the gust inputs in isotropic turbulence are specified in terms of a single quantity. The spectrum of β_g is selected for this purpose inasmuch as β_g is directly related to the linear gust component v_g for which the spectrum is available from turbulence theory. Therefore, the final form of the equations is given by

$$\begin{Bmatrix} \Phi_\phi \\ \Phi_\psi \\ \Phi_\beta \end{Bmatrix} = \begin{bmatrix} \left| \frac{\phi}{D\phi_g} \right|^2 & \left| \frac{\phi}{D\psi_g} \right|^2 & \left| \frac{\phi}{\beta_g} \right|^2 \\ \left| \frac{\psi}{D\phi_g} \right|^2 & \left| \frac{\psi}{D\psi_g} \right|^2 & \left| \frac{\psi}{\beta_g} \right|^2 \\ \left| \frac{\beta}{D\phi_g} \right|^2 & \left| \frac{\beta}{D\psi_g} \right|^2 & \left| \frac{\beta}{\beta_g} \right|^2 \end{bmatrix} \begin{Bmatrix} \left| \frac{D\phi_g}{\beta_g} \right|^2 \\ \left| \frac{D\psi_g}{\beta_g} \right|^2 \\ 1 \end{Bmatrix} \Phi_{\beta_g} \quad (84)$$

GUST SPECTRA AND THEIR RELATIONSHIPS

The following relationships for the gust spectra and gust velocities are based on the assumptions of homogeneous isotropic turbulence:

$$\left. \begin{aligned} \Phi_{u_g}(\omega) &= \Phi_{v_g}(\omega) = U^2 \Phi_{\beta_g}(\omega) \\ \overline{u_g^2} &= \overline{v_g^2} = \overline{w_g^2} \end{aligned} \right\} \quad (85)$$

Physically, these relations state that, regardless of the motions or the direction of travel of the airplane, the vertical- and side-gust components measured at the same point on the airplane have the same spectrum and that the mean-square value of all three gust components is the same.

For the purposes of equation (84), the power spectra of gust gradients as a function of the gust spectra measured at one point are required in the form of the ratios of the power spectra of $D\phi_g$ and $D\psi_g$ to the power spectrum of β_g when the latter is known. Taking the square of the absolute value of equation (72) and considering the equation to apply at any given frequency yields

$$\left. \begin{aligned} |D\phi_g|^2 &= \frac{4}{(C_{lp})_w^2} |C_l(w_g)|^2 \\ \Phi_{D\phi_g} &= \frac{4}{(C_{lp})_w^2} \Phi_{C_l(w_g)} \end{aligned} \right\} \quad (86)$$

Dividing both sides of equations (86) by the power spectrum of the side gust or vertical gust (see eqs. (85)) as measured at the center of gravity gives

$$\frac{\Phi_{D\phi_g}}{\Phi_{\beta_g}} = \frac{4U^2}{(C_{lp})_w^2} \frac{\Phi_{C_l(w_g)}}{\Phi_{w_g}}$$

At any given frequency, then

$$\left| \frac{D\phi_g}{\beta_g} \right|^2 = \frac{4U^2}{(C_{lp})_w^2} \frac{\Phi_{C_l(w_g)}}{\Phi_{w_g}} \quad (87)$$

Expressions for both power spectra on the right-hand side of equation (22) are available in the literature. An expression based on measurements of turbulence in wind tunnels (ref. 19) which appears to fit well most of the available data from measurements of atmospheric turbulence (for example, refs. 18 and 23) is given in terms of β_g by

$$U^2 \Phi_{\beta_g} = \Phi_{w_g} = \frac{w_g^2 L}{\pi U^3} \frac{1+3k'^2}{(1+k'^2)^2} \quad (88)$$

where $k' = \omega L/U$ and L is the so-called scale of turbulence. A plot of equation (88) is given in figure 14.

Calculations based on analytical expressions (similar to eq. (88)) for the gust spectra measured at one point are given in reference 12 for the power spectra of the coefficients of the rolling and yawing moments on wings of arbitrary span which are subject to continuous isotropic turbulence. These spectra, which take into account the random distributions of gusts across the span and along the flight path, are given in reference 12 for various values of $\beta' = b/L$ and for four spanwise lift distributions on the wing. Since the effect of different lift distributions was small, only one distribution (the rectangular distribution) is considered and the power spectrum of rolling-moment coefficient due to vertical gusts on the wing is denoted here by

$$\Phi_{C_{l(w_g)}} = \frac{\overline{w_g^2} L (C_{lp})_w^2}{\pi U^3} \times \text{Quantity plotted in fig. 7(a), ref. 12}$$

The quantity plotted in figure 7(a) of reference

12 is $\frac{\Phi_{C_l(k')}}{w_g^2 L C_{lp}^2 U^3 \pi}$. Dividing the above equation

by equation (88) and substituting the result into equation (87) yields

$$\left| \frac{D\phi_g}{\beta_g} \right|^2 = 4 \times \frac{\text{Quantity plotted in fig. 7(a), ref. 12}}{\frac{1+3k'^2}{(1+k'^2)^2}} \quad (89)$$

This relationship is plotted in figure 15 for the various values of β' used in reference 12 and as a function of reduced frequency ω' , where

$$\omega' = \frac{\omega b}{U} = \beta' k' \quad (90)$$

In a like manner the relationship between the power spectra of yawing gusts and side gusts may be determined. From equation (76),

$$\Phi_{D\psi_g} = \frac{4}{(C_{lr})_w^2} \Phi_{C_l(w_g)} \quad (91)$$

Dividing through by the power spectrum of the

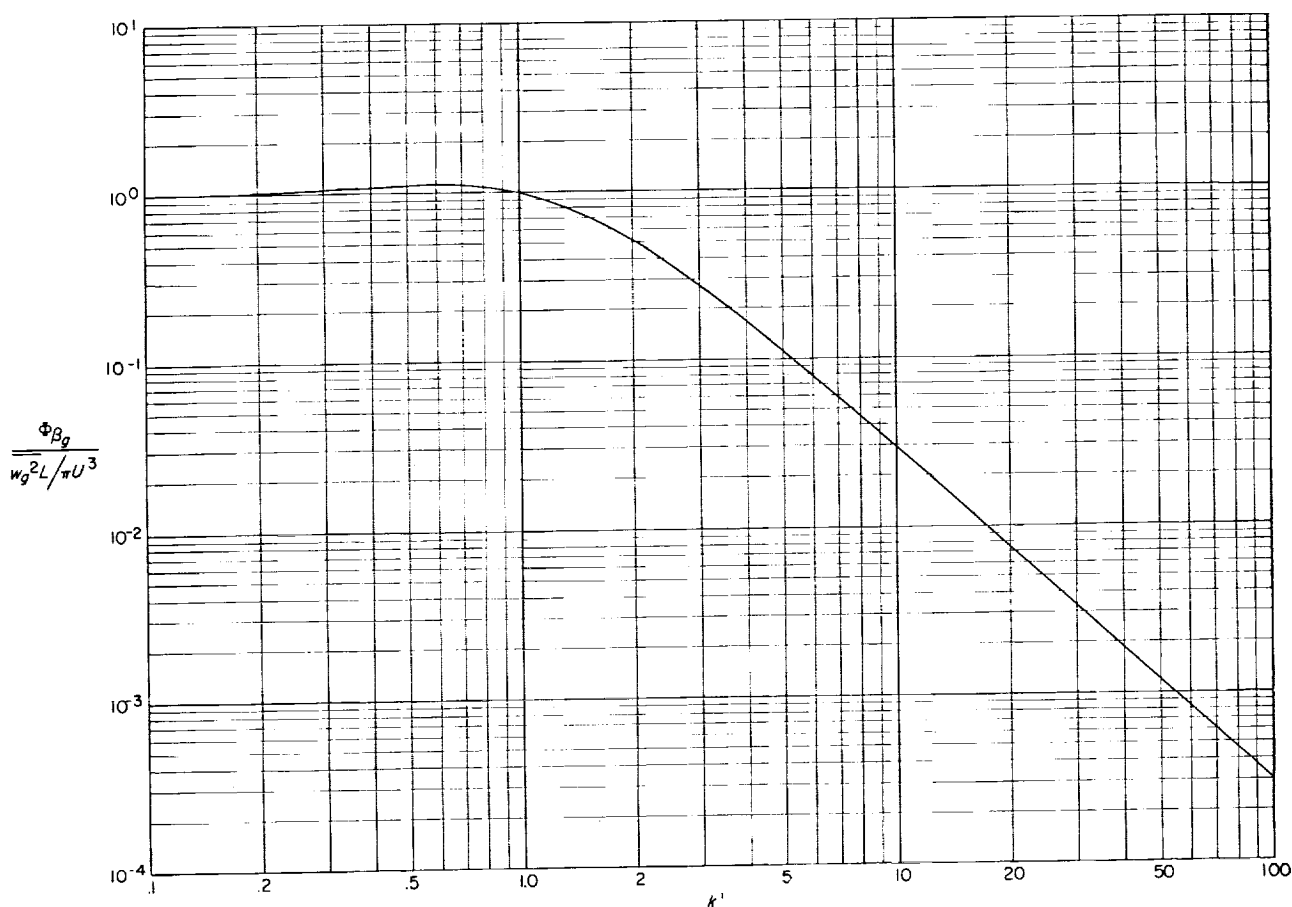


FIGURE 14.—Power spectral density of the side-gust component of isotropic turbulence. $k' = \frac{\omega L}{U}$.

side gusts or vertical gusts as measured at the center of gravity gives

$$\frac{\Phi_{D\psi_g}}{\Phi_{\beta_g}} = \frac{4U^2}{(C_{lr})_w^2} \frac{\Phi_{C_l(u_g)}}{\Phi_{w_g}}$$

where at any given frequency

$$\left| \frac{D\psi_g}{\beta_g} \right|^2 = \frac{4U^2}{(C_{lr})_w^2} \frac{\Phi_{C_l(u_g)}}{\Phi_{w_g}} \quad (92)$$

Again, from reference 12, the power spectrum of the rolling-moment coefficient due to horizontal gusts on the wing is given by

$$\Phi_{C_l(u_g)} = \frac{\overline{u_g^2} L (C_{lp})_w^2 \alpha_w^2}{\pi U^3} \times \text{Quantity plotted}$$

in fig. 9(a), ref. 12

The quantity plotted in figure 9(a) of reference 12 is $\frac{\Phi_{C_l}(k')}{\overline{u_g^2} L C_{lp} \alpha_0^2 / U^3 \pi}$, where α_0 is trim angle of attack.

Dividing the above equation by equation (88) and substituting the result into equation (92) yields the required relationship between the yawing- and side-gust spectra:

$$\left| \frac{D\psi_g}{\beta_g} \right|^2 \left(\frac{\alpha C_{lp}}{C_{lr}} \right)_w^{-2} = 4 \times \frac{\text{Quantity plotted in fig. 9(a), ref. 12}}{\frac{1+3k'^2}{(1+k'^2)^2}} \quad (93)$$

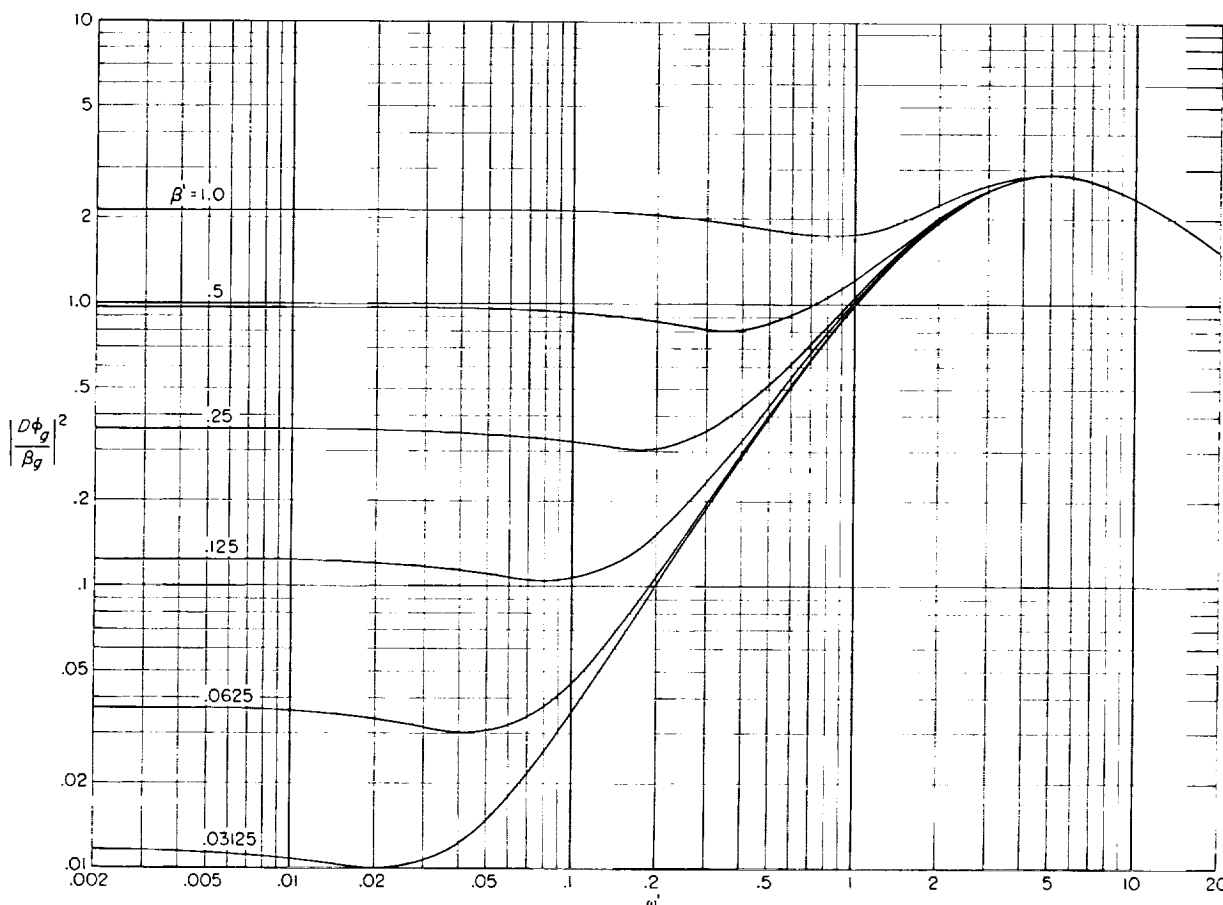


FIGURE 15.—Ratio of rolling-gust power spectrum to side-gust power spectrum as a function of reduced frequency ω' for various values of β' , the ratio of wing span to scale of turbulence.

This relationship is plotted in figure 16 as a function of ω' for a range of values of β' .

Presentation of the data of figures 15 and 16 in the form of ratios between the gust spectra is not meant to imply that these data are independent of the approximate spectra chosen for β_g . These ratios represent the filtering effect of the wing on gusts having the frequency spectrum given by equation (88), and, if other approximations to this frequency spectrum were used, the filtering effect would not necessarily be the same.

DISCUSSION

The foregoing development leads to a set of equations which hold for small disturbances about some trimmed flight condition. In the application of the method, it is necessary to know the flight conditions, the stability derivatives of the airplane at those flight conditions, and some basic

physical dimensions of the airframe. Such quantities are required for any dynamics study of an airplane and, aside from the equations and figures given herein, no additional information is required to obtain the lateral response of the airplane in power-spectral form to continuous atmospheric turbulence.

As an aid in setting up the required calculations, a list of columns and steps which may be followed in the calculation of Φ_ϕ is given in table III. The column numbers and headings are listed vertically in this table; however, they would be arranged horizontally across the top of an actual calculation sheet. Although the calculation of Φ_ϕ is used as an example, the same steps are followed for the response of the airplane in any angular displacement, rate, or acceleration. For a given airplane and flight condition the first 12 steps (part (a)) may be tabulated from the equations and plots of this part of the report. Steps 13 to 15 (part (b)) are

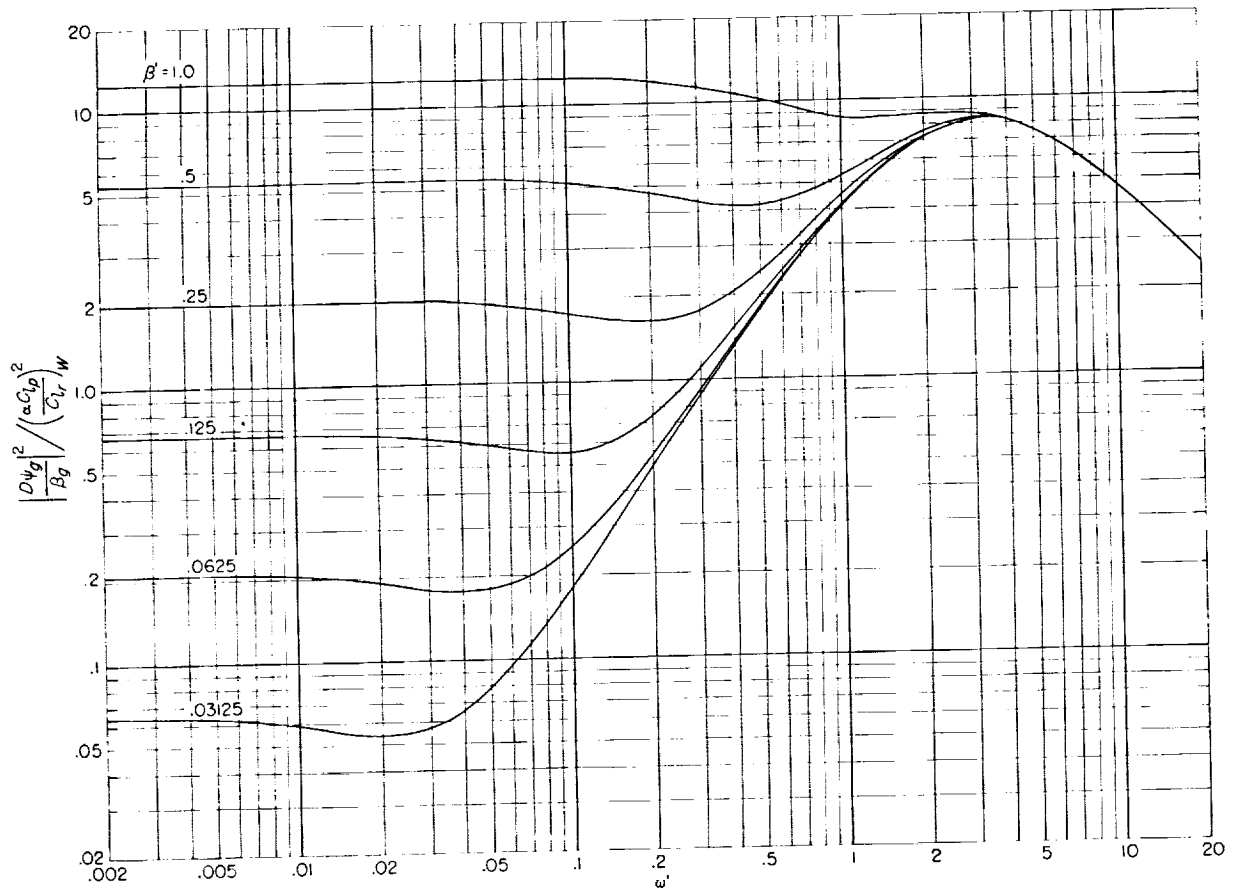


FIGURE 16.—Curves for determining the ratio of yawing-gust power spectrum to side-gust power spectrum for various values of the ratio of wing span to scale of turbulence β' , wing angle of attack α_w , and wing stability derivatives C_{L_v} and C_{L_p} .

used for the calculation of the response of the airplane in any lateral angular displacement (for the example, ϕ is used). It is necessary to tabulate the real and imaginary parts of three transfer functions relating C_b , C_n , and C_Y to that response parameter (columns ⑬ to ⑮) and to follow the indicated steps. For each of the other response motions or their derivatives it is only necessary to substitute the appropriate transfer functions into columns ⑬ to ⑮ and to repeat the process.

In calculating the frequency-dependent stability derivatives (columns ④ to ⑨), care should be taken to insure that, when $\omega=0$, these derivatives agree with the steady-state aerodynamic derivatives used in calculating the transfer functions of equation (80).

The method requires a choice of values for the

scale of turbulence L and the mean square of the turbulence $\overline{w_g^2}$. Very little information is available at present on the proper magnitude for L ; however, it appears to be within the range of 1,000 to 2,000 feet and probably closer to the lower figure. The mean square of the turbulence depends on the severity of the turbulence to be considered. As an approximation, a value of $\overline{w_g^2}$ of $(3 \text{ ft/sec})^2$ for light turbulence, $(6 \text{ ft/sec})^2$ for moderate turbulence, and $(10 \text{ ft/sec})^2$ for thunderstorms may be used. (See ref. 23.)

In the present method the effects of turbulence on the airplane have been separated into the equivalent effects of rolling, yawing, and sideslip of the airplane in still air. The application of this concept, however, has been made in such a way that the effects of the u -, v -, and w -components of the gusts are taken into account as was

TABLE III.—SAMPLE CALCULATION PROCEDURE

(a) Calculation of frequency-dependent derivatives and gust spectra

Column number	Column heading	Instructions
①	ω	
②	ω'	$= \omega \frac{b}{U}$
③	k'	$= \omega' / \beta'$
④	$R[(C_{lp})_{WT}]$	From equation (C1)
⑤	$I[(C_{lp})_{WT}]$	
⑥	$R[(C_{n\beta})_{FT}]$	
⑦	$I[(C_{n\beta})_{FT}]$	
⑧	$R[(C_{Y\beta})_{FT}]$	
⑨	$I[(C_{Y\beta})_{FT}]$	From equation (C2)
⑩	$ D\phi_s/\beta_s ^2$	Quantity plotted in fig. 15
⑪	$ D\psi_s/\beta_s ^2$	Quantity plotted in fig. 16 $\times \left(\frac{\alpha C_{lp}}{C_{lr}}\right)_w^2$
⑫	Φ_{β_s}	Quantity plotted in fig. 14 $\times \frac{\overline{w_s^2} L}{\pi \ell^3}$

(b) Calculation of roll response

Column number	Column heading	Instructions
⑬	$R(\phi/C_l)$	Evaluation of airframe transfer functions (eqs. 26 to 35) at specified values of ω'
⑭	$I(\phi/C_l)$	
⑮	$R(\phi/C_n)$	
⑯	$I(\phi/C_n)$	
⑰	$R(\phi/C_Y)$	
⑱	$I(\phi/C_Y)$	
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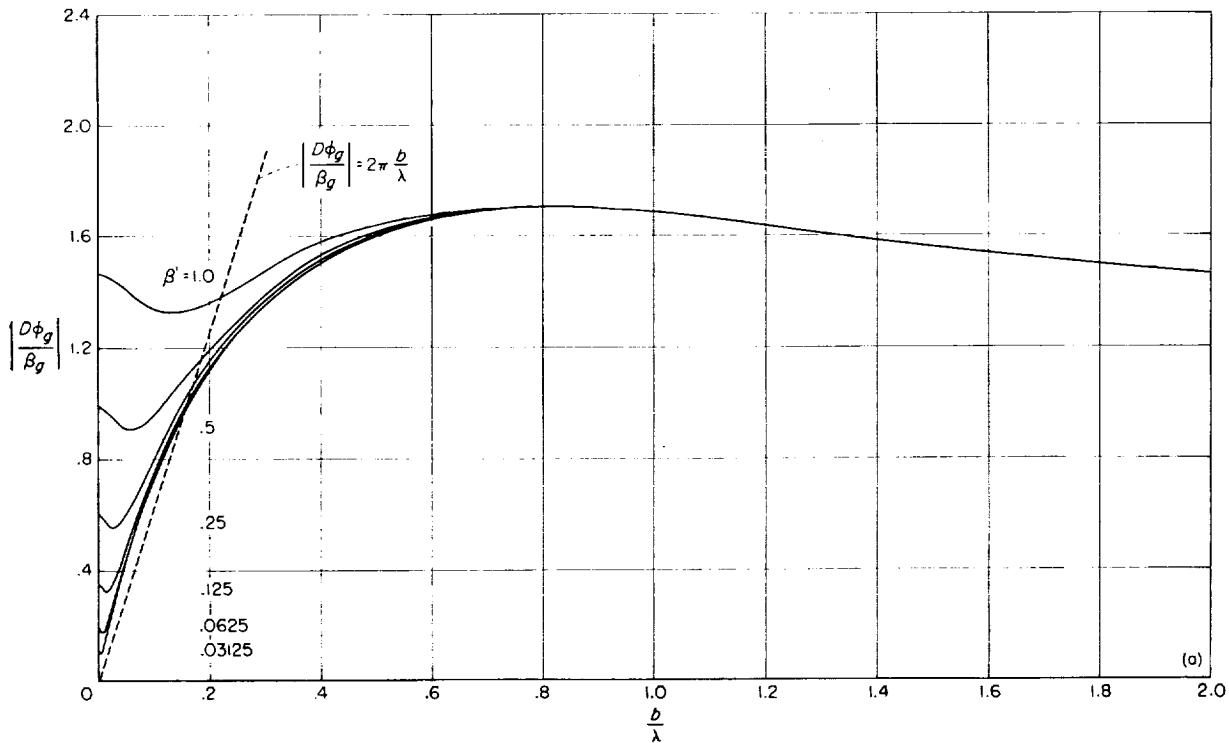
done in part I. Although the procedure of part II differs from that of part I, exactly the same effects are treated in both methods and both methods yield the same results. This agreement has been verified by using the present method to calculate responses for the example airplanes of part I. Since the responses of three airplanes are presented in part I, no numerical examples are presented in part II.

One distinction should be noted between the present method and the more conventional treatment as contained in equation (68). In the conventional treatment, yawing gusts are assumed to include both the rotational effects introduced by gradients of side gust along the fuselage and by gradients of horizontal gusts across the wing span. Thus, the values of $C_{\dot{\gamma}}$, $C_{n_{\dot{\gamma}}}$, and $C_{Y_{\dot{\gamma}}}$ of equation (70) would be those for the entire airplane. In the present method, the yawing gusts are assumed to include only the effects of horizontal gusts on the wing, and the values of $(C_{\dot{\gamma}})_w$ and $(C_{n_{\dot{\gamma}}})_w$ used in equation (78) are those for the wing alone. Rotational effects of the side gusts along the fuselage are completely accounted for by the complex stability derivatives $(C_{\dot{\beta}})_{WT}$, $(C_{n_{\dot{\beta}}})_{FT}$, and $(C_{Y_{\dot{\beta}}})_{FT}$. This method is used because it allows the utilization of the more accurate calculation of the fuselage penetration effects given in reference 11 and because the separation of the effects of the wing and fuselage allows the random distribution of horizontal gusts across the span to be taken into account.

With most current airplane configurations in flight at low angles of attack a simplification of equations may be obtained by neglecting the yawing gusts on the wing. In the numerical examples of part I it was observed that this gust component had a negligible effect in all degrees of freedom for the airplanes investigated. This result is the basis for the assumption made in the derivation of the method used herein that the cross power between the gust-velocity components may be neglected. By the theory of isotropic turbulence only the cross power between the components herein referred to as yawing gusts and side gusts exists. However, when the yawing-gust contribution to the motion of the airplane is small, neglecting this cross power appears to be justified.

It was also found in part I that the further simplification of neglecting the contribution of both rolling and yawing gusts in calculating the response in sideslip is justified.

The plots of the ratios of rolling gusts and yawing gusts to side gusts give a physical picture of the relative importance of these gust disturbances at various frequencies. For calculation purposes, the plots of the quantities $\left| \frac{D\phi_g}{\beta_g} \right|^2$ and $\left| \frac{D\psi_g}{\beta_g} \right|^2$ as functions of frequency on log-log paper (figs. 15 and 16) are convenient, but a plot of this type gives a somewhat distorted picture of the true variations of these quantities. For this reason, plots of $\left| \frac{D\phi_g}{\beta_g} \right|$ and $\left| \frac{D\psi_g}{\beta_g} \right|$ as functions of frequency on linear scales are given in figure 17. Frequency is plotted in terms of the ratio of wing span to gust wavelength b/λ . These curves show that, for the small values of β' (ratios of wing span to scale of turbulence) ordinarily encountered, the ratio $\left| \frac{D\phi_g}{\beta_g} \right|$ or $\left| \frac{D\psi_g}{\beta_g} \right|$ falls on a single curve as a function of b/λ , except at low values of b/λ . The curves reach a peak in the neighborhood of $b/\lambda = 0.5$ to 1.0, as might be expected on the basis that the spanwise averaging effects would become important when the gust wavelength is shorter than the span. The hypothesis, expressed in reference 8, that the gust gradient measured along the flight path would give an approximation to the effective spanwise gust gradient is indicated by the dashed line drawn on the figure (fig. 17(a)). The agreement in the trends of the curves with this dashed line shows that this hypothesis may be a reasonable explanation of the mechanism of the rolling and yawing effects of turbulence over an intermediate range of wavelengths. At shorter wavelengths, the spanwise averaging causes a decrease in the effects; whereas at long wavelengths, or low frequencies, another mechanism apparently comes into play to increase the rolling and yawing effects. This mechanism is believed to be the chance encounter of the wing with rolling and yawing gusts distributed along the flight path at relatively long intervals. This effect increases with the ratio of wing span to scale of turbulence. Possibly this result occurs because a wing of larger span effectively samples a larger portion of the atmosphere and is therefore more likely to encounter rolling and yawing gusts. This effect might be



(a) Ratio of the rolling-gust component to the side-gust component. The approximation based on a constant anti-symmetric gust-velocity gradient over the wing span is indicated by the dashed line.

FIGURE 17.—Ratio of the absolute amplitudes of the rolling- and yawing-gust components to the side-gust component as a function of the ratio of wing span to gust wavelength.

important in explaining increased lateral-control difficulties of large airplanes during landing approaches in rough air, if it can be shown that the scale of turbulence decreases at low altitudes.

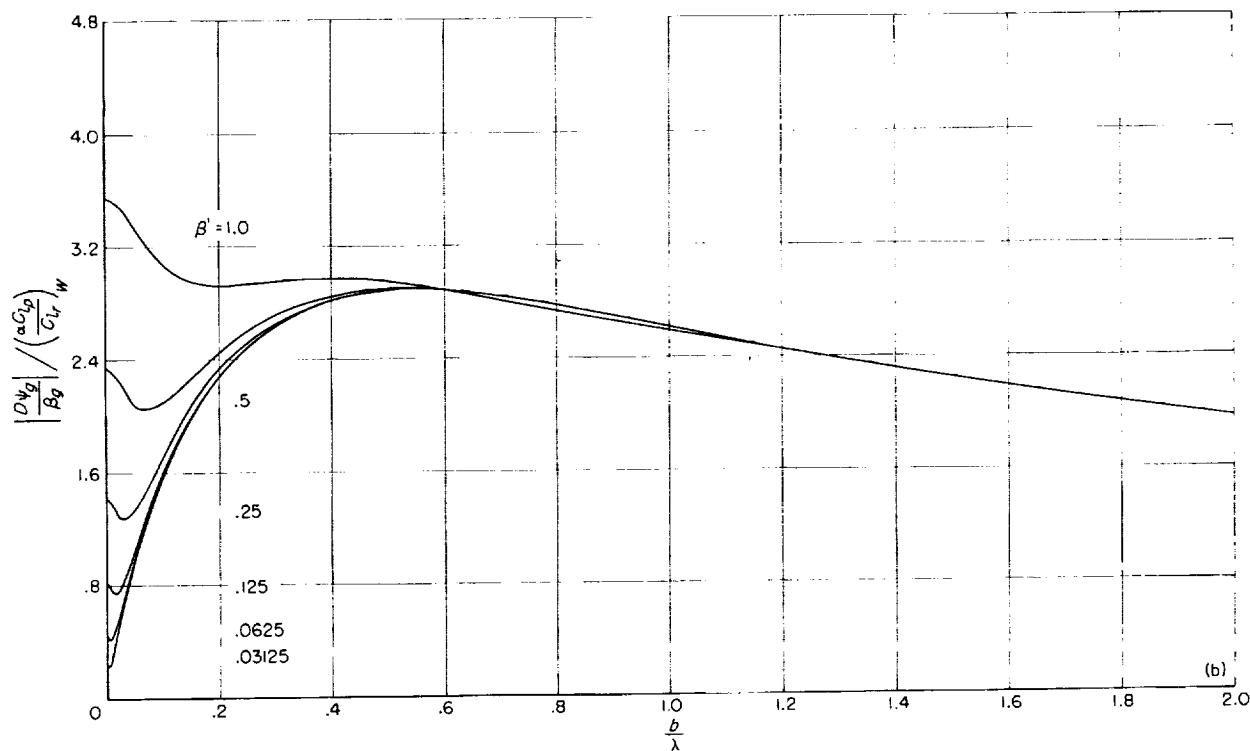
CONCLUDING REMARKS

By resolving the gust velocities of random isotropic atmospheric turbulence into three components aligned with the stability axes of an airplane, the lateral forces and moments on the airplane and the resulting lateral motions of the airplane have been derived. The method involves the computation of the statistical forces and moments experienced by the airplane due to isotropic turbulence having random variations along the fuselage and across the wing span. On the basis of these gust-induced forces and moments referenced to the center of gravity of the airplane, the power spectral response of the airplane in any lateral degree of freedom (be it displacement, velocity, or acceleration) may be computed by using the appropriate transfer functions of the airplane. Since the gust

velocities, the forces, and the moments are defined along the flight path in terms of their statistical average, the resulting motions are defined in terms of their power spectra, the mean-square value of the gust velocities, and the scale of atmospheric turbulence.

By using this method, calculations have been made on three airplanes, which differed in size, experiencing continuous atmospheric turbulence while in a trimmed condition of low angle of attack. On the basis of these examples, certain possible simplifications to the equations have been brought out. Primary among these are the negligible effect of horizontal gusts u_g and the minor contribution of side force in computing the motions. In general, then, the lateral motions at low trim angles of attack are due primarily to the ability of the side gusts v_g and vertical gusts w_g to produce yawing and rolling moments on the airplane.

With regard to lateral motions the method used in part I for considering random variations of the gust velocities along the fuselage and across the



(b) Ratio of the yawing-gust component to the side-gust component.

FIGURE 17.—Concluded.

span is a concept hitherto only approximated with constant gradients of gust velocity or completely ignored. Such approximations do not appear to be justified for gusts having wavelengths shorter than twice the span or fuselage length of the airplane. Although these less comprehensive methods correctly predict the trends in the vicinity of the Dutch roll mode of the airplane, they lead to the erroneous conclusion that the power spectra of the lateral motion due to the unsymmetrical components of the gust across the span ($\partial w_g / \partial y$ and $\partial u_g / \partial y$) fall off to zero at wavelengths greater than the integral scale of the turbulence. The magnitude of difference between the methods increases with frequency above the Dutch roll mode since in this general region of frequencies the short-period (with respect to the size of the airplane) random variations of gust velocity along

the fuselage and across the span have a predominant effect on the moments and motions of the airplane.

In part II a procedure is presented for calculating in power-spectral form the lateral response of airplanes to random atmospheric turbulence. By following the tabulated sample calculation procedure, these calculations may be made in a routine manner without detailed knowledge of the derivation of the method. It has been verified that the method of part I and the method of part II give identical results. The method of part II requires simpler calculations and provides a clearer physical picture of the relations between the various sources of lateral gust disturbances.

LANGLEY RESEARCH CENTER,
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION,
LANGLEY FIELD, VA., July 23, 1959.

APPENDIX A

PROOF OF POWER SPECTRAL AND CROSS-POWER SPECTRAL RELATIONSHIPS

The physical and theoretical aspects of statistical methods of analyzing dynamic systems (generalized harmonic analysis) have been treated in a number of excellent papers. (Among these are refs. 13, 14, and 21.) The following definitions and proofs are purely a mathematical means of utilizing the statistical concept and should be treated as useful supplements to the referenced statistical theory.

Consider the simple equation

$$\eta = \frac{\eta}{\delta} \delta + \frac{\eta}{\nu} \nu \quad (\text{A1})$$

where η , δ , and ν are complex variables and η/δ and η/ν are complex transfer functions such that

$$\left. \begin{aligned} \eta &= R(\eta) + iI(\eta) & \frac{\eta}{\delta} &= \left| \frac{\eta}{\delta} \right| e^{i\phi(\eta/\delta)} \\ \delta &= R(\delta) + iI(\delta) & \frac{\eta}{\nu} &= \left| \frac{\eta}{\nu} \right| e^{i\phi(\eta/\nu)} \\ \nu &= R(\nu) + iI(\nu) \end{aligned} \right\} \quad (\text{A2})$$

With the use of an asterisk to denote the conjugate of a complex quantity, it is also true that

$$\eta^* = \left(\frac{\eta}{\delta} \right)^* \delta^* + \left(\frac{\eta}{\nu} \right)^* \nu^* \quad (\text{A3})$$

Thus

$$\begin{aligned} \eta \eta^* &= \frac{\eta}{\delta} \left(\frac{\eta}{\delta} \right)^* \delta \delta^* + \frac{\eta}{\nu} \left(\frac{\eta}{\nu} \right)^* \nu \nu^* + \frac{\eta}{\delta} \left(\frac{\eta}{\nu} \right)^* \delta \nu^* + \frac{\eta}{\nu} \left(\frac{\eta}{\delta} \right)^* \delta^* \nu \\ &= \left| \frac{\eta}{\delta} \right|^2 |\delta|^2 + \left| \frac{\eta}{\nu} \right|^2 |\nu|^2 + \frac{\eta}{\delta} \left(\frac{\eta}{\nu} \right)^* \delta \nu^* + \frac{\eta}{\nu} \left(\frac{\eta}{\delta} \right)^* \delta^* \nu \end{aligned} \quad (\text{A4})$$

By expansion in terms of equations (A2) it may be seen that

$$\begin{aligned} \delta \nu^* &= (\delta^* \nu)^* \\ \frac{\eta}{\delta} \left(\frac{\eta}{\nu} \right)^* &= \left[\left(\frac{\eta}{\delta} \right)^* \frac{\eta}{\nu} \right]^* \end{aligned}$$

and, in fact,

$$\frac{\eta}{\delta} \left(\frac{\eta}{\nu} \right)^* \delta \nu^* = \left[\left(\frac{\eta}{\delta} \right)^* \frac{\eta}{\nu} \delta^* \nu \right]^* \quad (\text{A5})$$

Since the sum of a complex number and its conjugate is equal to twice the real part, that is,

$$(R + iI) + (R - iI) = 2R$$

then the substitution of equation (A5) into equation (A4) yields

$$|\eta|^2 = \left| \frac{\eta}{\delta} \right|^2 |\delta|^2 + \left| \frac{\eta}{\nu} \right|^2 |\nu|^2 + 2R \left[\left(\frac{\eta}{\delta} \right)^* \frac{\eta}{\nu} \delta^* \nu \right] \quad (\text{A6})$$

It is equally true to write

$$2R \left[\left(\frac{\eta}{\delta} \right)^* \frac{\eta}{\nu} \delta^* \nu \right] = 2R \left[\frac{\eta}{\delta} \left(\frac{\eta}{\nu} \right)^* \delta \nu^* \right] \quad (\text{A7})$$

so that either form may be used.

Dividing both sides of equation (A6) by T and taking the limit of the statistical variables as T approaches infinity give the form

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} |\eta|^2 &= \left| \frac{\eta}{\delta} \right|^2 \lim_{T \rightarrow \infty} \frac{1}{T} |\delta|^2 + \left| \frac{\eta}{\nu} \right|^2 \lim_{T \rightarrow \infty} \frac{1}{T} |\nu|^2 \\ &\quad + 2R \left[\left(\frac{\eta}{\delta} \right)^* \frac{\eta}{\nu} \lim_{T \rightarrow \infty} \frac{1}{T} \delta^* \nu \right] \end{aligned} \quad (\text{A8})$$

where, by the definitions of equations (5) and (6) of the text, equation (A8) is equivalent to

$$\Phi_\eta = \left| \frac{\eta}{\delta} \right|^2 \Phi_\delta + \left| \frac{\eta}{\nu} \right|^2 \Phi_\nu + 2R \left[\left(\frac{\eta}{\delta} \right)^* \frac{\eta}{\nu} \Phi_{\delta\nu} \right] \quad (\text{A9})$$

If the two variables ν and δ are statistically independent, their cross power is zero. For equations with more than two variables the process is identical and, of course, yields more terms. For an equation whose solution is a function of m variables, there will result m power-spectrum

terms and $\sum_{n=1}^m (n-1)$ cross-power terms in the form of equation (A9), or in the equivalent form

$$\Phi_\eta = \left| \frac{\eta}{\delta} \right|^2 \Phi_\delta + \left| \frac{\eta}{\nu} \right|^2 \Phi_\nu + 2R \left[\frac{\eta}{\delta} \left(\frac{\eta}{\nu} \right)^* \Phi_{\nu\delta} \right] \quad (\text{A10})$$

based on the identity of equation (A7).

For the cross spectrum between two statistical variables not appearing in the same equation, consider the equations

$$\eta = \frac{\eta}{\delta} \delta + \frac{\eta}{\nu} \nu$$

and

$$\epsilon = \frac{\epsilon}{\delta} \delta + \frac{\epsilon}{\nu} \nu \quad (\text{A11})$$

where the complex variable ϵ , in addition to the definitions of equation (A2), is defined by

$$\epsilon = R(\epsilon) + iI(\epsilon) \quad (\text{A12})$$

Multiplying equation (A3) by equation (A11) gives

$$\eta^* \epsilon = \left(\frac{\eta}{\delta} \right)^* \frac{\epsilon}{\delta} \delta^* \delta + \left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\nu} \nu^* \nu + \left(\frac{\eta}{\delta} \right)^* \frac{\epsilon}{\nu} \delta^* \nu + \left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\delta} \nu^* \delta \quad (\text{A13})$$

where the identity

$$\left(\frac{\eta}{\delta} \right)^* \frac{\epsilon}{\nu} \delta^* \nu \equiv \left[\frac{\eta}{\delta} \left(\frac{\epsilon}{\nu} \right)^* \delta \nu^* \right]^* \quad (\text{A14})$$

may be proved by the use of equations (A2) and

(A11), so that

$$\eta^* \epsilon = \left(\frac{\eta}{\delta} \right)^* \frac{\epsilon}{\delta} |\delta|^2 + \left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\nu} |\nu|^2 + 2R \left[\left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\delta} \nu^* \delta \right] \quad (\text{A15})$$

Dividing both sides by T and taking the limit of the statistically dependent variables as T approaches infinity yield

$$\lim_{T \rightarrow \infty} \frac{1}{T} \eta^* \epsilon = \left(\frac{\eta}{\delta} \right)^* \frac{\epsilon}{\delta} \lim_{T \rightarrow \infty} \frac{1}{T} |\delta|^2 + \left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\nu} \lim_{T \rightarrow \infty} \frac{1}{T} |\nu|^2 + 2R \left[\left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\delta} \lim_{T \rightarrow \infty} \nu^* \delta \right] \quad (\text{A16})$$

By the definition of equations (5) and (6) of the text, equation (A16) becomes

$$\Phi_{\eta\epsilon} = \left(\frac{\eta}{\delta} \right)^* \frac{\epsilon}{\delta} \Phi_\delta + \left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\nu} \Phi_\nu + 2R \left[\left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\delta} \Phi_{\nu\delta} \right] \quad (\text{A17})$$

where again the identity

$$2R \left[\left(\frac{\eta}{\nu} \right)^* \frac{\epsilon}{\delta} \Phi_{\nu\delta} \right] \equiv 2R \left[\frac{\eta}{\nu} \left(\frac{\epsilon}{\delta} \right)^* \Phi_{\delta\nu} \right]$$

may be used in equation (A17) when desired. As before, $\Phi_{\nu\delta} = \Phi_{\delta\nu} = 0$ where ν and δ are statistically independent; but, in general, where the two equations are both functions of m variables, there will result m power-density terms and $\sum_{n=1}^m (n-1)$ cross-power terms.

APPENDIX B

RELATIONSHIP BETWEEN POWER SPECTRA OF AIRPLANE FORCES AND MOMENTS AND POWER SPECTRA OF GUST VELOCITIES

Consider the system defined by equation (2) of the text, which is rewritten here for convenience in the matrix form

$$\{C_j\} = [a_{jm}]\{u_m\} \quad (B1)$$

where C_j is a force or moment coefficient (C_L , C_n , or C_Y) and a_{jm} is a transfer function relating C_j to u_m . The gust velocities u_1 , u_2 , and u_3 (u_m where $m=1, 2$, or 3) are the three orthogonal velocity components of isotropic turbulence, and the cross power between any two components is zero; that is,

$$\Phi_{u_1 u_2} = \Phi_{u_2 u_3} = \Phi_{u_3 u_1} = 0 \quad (B2)$$

In the power-spectral domain (see appendix A) the first equation of the system is defined by

$$\begin{aligned} \Phi_{C_1} = & |a_{11}|^2 \Phi_{u_1} + |a_{12}|^2 \Phi_{u_2} + |a_{13}|^2 \Phi_{u_3} \\ & + 2R(a_{11}^* a_{12} \Phi_{u_1 u_2} + a_{12}^* a_{13} \Phi_{u_2 u_3} + a_{13}^* a_{11} \Phi_{u_3 u_1}) \end{aligned} \quad (B3)$$

All the cross-power terms of equation (B3) are zero. The other equations yield similar results; therefore, the system is defined by

$$\{\Phi_{C_j}\} = [|a_{jm}|^2] \{\Phi_{u_m}\} \quad (B4)$$

which is the result given by equation (9) of the text.

Now consider the cross-power terms of the C_j matrix, where, for instance,

$$\begin{aligned} \Phi_{C_1 C_2} &= \lim_{T \rightarrow \infty} \frac{1}{T} C_1^* C_2 \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \left[\sum_{m=1}^3 a_{1m} u_m \right]^* \left[\sum_{m=1}^3 a_{2m} u_m \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{m=1}^3 a_{1m}^* u_m^* \sum_{m=1}^3 a_{2m} u_m \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \left(\sum_{m=1}^3 a_{1m}^* a_{2m} u_m^* u_m + 6 \text{ cross-power terms involving } u_1^* u_2, u_2^* u_3, \text{ and so forth} \right) \end{aligned} \quad (B5)$$

In the limit, then

$$\begin{aligned} \Phi_{C_1 C_2} &= \sum_{m=1}^3 a_{1m}^* a_{2m} \lim_{T \rightarrow \infty} \frac{1}{T} u_m^* u_m + 0 \\ &= \sum_{m=1}^3 a_{1m}^* a_{2m} \Phi_{u_m} \end{aligned} \quad (B6)$$

where by definition

$$\Phi_{u_m} = \lim_{T \rightarrow \infty} \frac{1}{T} u_m^* u_m = \lim_{T \rightarrow \infty} \frac{1}{T} |u_m|^2 \quad (B7)$$

A similar derivation for the other two cross-power combinations yields similar results; therefore, in

general,

$$\{\Phi_{C_j C_k}\} = [a_{jm}^* a_{km}] \{\Phi_{u_m}\} \quad (j \neq k) \quad (\text{B8a})$$

where C_k is also a force or moment coefficient and a_{km} is a transfer function relating C_k to u_m , and

$$\{\Phi_{C_j C_k}\} = [|a_{jm}|^2] \{\Phi_{u_m}\} \quad (j = k) \quad (\text{B8b})$$

Equation (B8b) is seen to be identical to equation (B4). In expanded form, for the case considered

$$\begin{Bmatrix} \Phi_{C_1 C_2} \\ \Phi_{C_2 C_3} \\ \Phi_{C_3 C_1} \end{Bmatrix} = \begin{bmatrix} a_{11}^* a_{21} & a_{12}^* a_{22} & a_{13}^* a_{23} \\ a_{21}^* a_{31} & a_{22}^* a_{32} & a_{23}^* a_{33} \\ a_{31}^* a_{11} & a_{32}^* a_{12} & a_{33}^* a_{13} \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \quad (\text{B9})$$

In the terms of equation (2) of the text, this result is equivalent to equation (10) of the text. Furthermore, other equally correct arrangements are possible, since

$$\{\Phi_{C_k C_j}\} = \{\Phi_{C_j}^* C_k\} \quad (\text{B10a})$$

$$\{\Phi_{C_k C_j}\} = [a_{jm} a_{km}^*] \{\Phi_{u_m}\} \quad (j \neq k) \quad (\text{B10b})$$

$$\{\Phi_{C_k C_j}\} = [|a_{jm}|^2] \{\Phi_{u_m}\} \quad (j = k) \quad (\text{B10c})$$

Equation (B10c) is identical to equations (B4) and (B8b).

APPENDIX C

FREQUENCY-DEPENDENT STABILITY DERIVATIVES DUE TO SIDE GUSTS

All derivatives are referred to the center of gravity of the airplane. The coefficient of rolling moment due to side gusts on the wing and vertical tail is given by

$$[C_{l_\beta}(\omega)]_{WT} = (C_{l_\beta})_W + (C_{l_\beta})_T \frac{h}{b} \left(1 + \frac{\partial \sigma}{\partial \beta} \right) e^{-i} \left(\frac{l \omega}{U} \right) \quad (\text{C1})$$

The coefficients of side force and yawing moment due to side gusts along the fuselage and vertical tail (see ref. 11) are expressed by

$$[C_{Y_\beta}(\omega)]_{FT} = \frac{2\pi}{S} \left\{ \frac{2s_0^2}{k_0^2} [1 - (1 - ik_0) e^{ik_0}] + \left(\frac{s_1 - s_0}{k_2 - k_1} \right)^2 [e^{-ik_1} - (1 - ik_1 + ik_2) e^{-ik_2}] \right\} \quad (\text{C2})$$

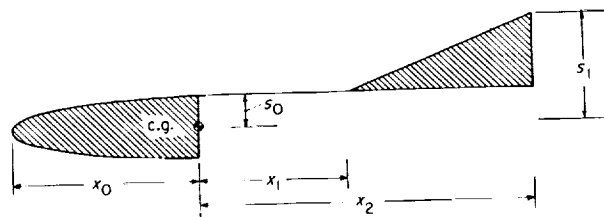
$$[C_{n_\beta}(\omega)]_{FT} = \frac{2\pi}{bS} \left\{ \frac{-2x_0 s_0^2}{k_0^3} [(2k_0 - ik_0^2 + i2) e^{ik_0} - i2] + \frac{(x_2 - x_1)(s_1 - s_0)^2}{(k_2 - k_1)^3} [(2k_2 - k_1 - i2 - ik_1 k_2 + ik_2^2) e^{-ik_2} - (k_1 - i2) e^{-ik_1}] \right\} \quad (\text{C3})$$

where

$$k_n = \frac{\omega x_n}{U} \quad (n=0,1,2)$$

and where x_0 , x_1 , x_2 , s_0 , and s_1 are the profile dimensions of the fuselage and vertical tail. These

profile dimensions, as given in the derivation in reference 11, are illustrated in the following sketch:



The fuselage is an ellipse truncated at the center of gravity, $2s_0$ being the length of the minor axis and $2x_0$ being the length of the major axis. The vertical tail is a right triangle of base $x_2 - x_1$ and height $s_1 - s_0$.

At $\omega = 0$, equations (C1), (C2), and (C3) become, respectively,

$$[C_{l_\beta}(\omega=0)]_{WT} = (C_{l_\beta})_W + (C_{l_\beta})_T \quad (\text{C4})$$

$$[C_{Y_\beta}(\omega=0)]_{FT} = -\frac{\pi}{S} [2s_0^2 + (s_1 - s_0)^2] \quad (\text{C5})$$

$$[C_{n_\beta}(\omega=0)]_{FT} = \frac{2\pi}{3Sb} \left[-2s_0^2 x_0 + (s_1 - s_0)^2 \left(x_2 + \frac{x_1}{2} \right) \right] \quad (\text{C6})$$

These quantities should be made numerically equal to the steady-state stability derivatives as obtained from flight-test measurements or more exact theories by suitably adjusting the dimensions of the assumed fuselage-tail profile.

APPENDIX D

TWO EQUIVALENT WAYS OF EXPRESSING THE POWER SPECTRAL RELATIONSHIP BETWEEN GUSTS AND AIRPLANE MOTIONS

The statement is made in the main text that, in the derivation of the power spectral relationships between the airplane motions and the gust-velocity inputs, the same result is obtained whether the matrix relationships are treated in the form

$$|\Delta^{-1}(\omega)|^2 |\tilde{G}(\omega)|^2 \quad (D1)$$

or in the form

$$|\Delta^{-1}\tilde{G}(\omega)|^2 \quad (D2)$$

In the former case, cross-power terms between the elements of the matrices (i.e., between the moments and forces on the airplane due to each gust component acting on the various parts of the airplane simultaneously) will appear, whereas in the latter case, these quantities do not appear explicitly but are taken into account when the matrices are first multiplied together. The former approach is used in the method of reference 1; the latter and simpler approach is used in the method of this paper. In order to show the equivalence of the two approaches it is necessary only to prove equality between equations (D1) and (D2), that is, that the square of the absolute value of the product of two matrices is equal to the product of the squares of their absolute values. Since matrix equations may be treated as linear algebraic quantities, the matrix equality defined in the main text

$$[\Delta(\omega)]^{-1}[\tilde{G}(\omega)] = [\Delta^{-1}\tilde{G}(\omega)] \quad (D3)$$

will be denoted by the complex quantities

$$AB = C \quad (D4)$$

The complex conjugate of this equality may be shown to be

$$A^*B^* = C^* \quad (D5)$$

Multiplying equations (D4) and (D5) yields

$$AA^*BB^* = CC^*$$

which is equivalent to

$$|A|^2|B|^2 = |C|^2$$

Hence,

$$|\Delta^{-1}(\omega)|^2 |\tilde{G}(\omega)|^2 = |\Delta^{-1}\tilde{G}(\omega)|^2 \quad (D6)$$

where the elements of these matrices are likewise complex quantities.

Insight into the difference between the two approaches may be shown by considering one element of the $[\Delta^{-1}\tilde{G}]$ matrix. Expanding equation (D3) by means of equations (78) and (80) of the main text gives

$$\begin{bmatrix} \frac{\phi}{C_i} & \frac{\phi}{C_n} & \frac{\phi}{C_y} \end{bmatrix} \begin{Bmatrix} \frac{1}{2}(C_{l_p})_w \\ \frac{1}{2}(C_{n_p})_w \\ 0 \end{Bmatrix} = D\phi_g$$

or, in its expanded form,

$$\frac{\phi}{D\phi_g} = \frac{1}{2}(C_{l_p})_w \frac{\phi}{C_i} + \frac{1}{2}(C_{n_p})_w \frac{\phi}{C_n} \quad (D7)$$

If the real and imaginary parts of equation (D7) are grouped, then the absolute value squared of equation (D7) becomes

$$\begin{aligned} \left| \frac{\phi}{D\phi_g} \right|^2 = & \left\{ \text{R} \left[\frac{1}{2}(C_{l_p})_w \frac{\phi}{C_i} + \frac{1}{2}(C_{n_p})_w \frac{\phi}{C_n} \right] \right\}^2 \\ & + \left\{ \text{I} \left[\frac{1}{2}(C_{l_p})_w \frac{\phi}{C_i} + \frac{1}{2}(C_{n_p})_w \frac{\phi}{C_n} \right] \right\}^2 \end{aligned} \quad (D8)$$

A different form involving cross-power terms may be obtained by writing the conjugate of equation (D7)

$$\frac{\phi}{D\phi_g} = \frac{1}{2} (C_{lp})_w \left(\frac{\phi}{C_l} \right)^* + \frac{1}{2} (C_{np})_w \left(\frac{\phi}{C_n} \right)^* \quad (D9)$$

and obtaining the product of equations (D7) and (D9):

$$\frac{\phi}{D\phi_g} \left(\frac{\phi}{D\phi_g} \right)^* = \frac{1}{4} (C_{lp})_w^2 \left(\frac{\phi}{C_l} \right)^* \left(\frac{\phi}{C_l} \right) + \frac{1}{4} (C_{np})_w^2 \left(\frac{\phi}{C_n} \right)^* \left(\frac{\phi}{C_n} \right) + \frac{1}{4} (C_{lp})_w (C_{np})_w \left[\left(\frac{\phi}{C_l} \right)^* \left(\frac{\phi}{C_n} \right) + \left(\frac{\phi}{C_n} \right)^* \left(\frac{\phi}{C_l} \right) \right] \quad (D10)$$

Since

$$\frac{\phi}{C_n} \left(\frac{\phi}{C_l} \right)^* = \left[\left(\frac{\phi}{C_n} \right)^* \frac{\phi}{C_l} \right]^*$$

and

$$\frac{\phi}{C_l} \left(\frac{\phi}{C_n} \right)^* + \left[\frac{\phi}{C_l} \left(\frac{\phi}{C_n} \right)^* \right]^* = 2R \left[\frac{\phi}{C_l} \left(\frac{\phi}{C_n} \right)^* \right]$$

equation (D10) may be written in the form

$$\left| \frac{\phi}{D\phi_g} \right|^2 = \frac{1}{4} (C_{lp})_w^2 \left| \frac{\phi}{C_l} \right|^2 + \frac{1}{4} (C_{np})_w^2 \left| \frac{\phi}{C_n} \right|^2 + 2R \left[\frac{1}{4} (C_{lp})_w (C_{np})_w \frac{\phi}{C_l} \left(\frac{\phi}{C_n} \right)^* \right] \quad (D11)$$

The term $2R \left[\frac{1}{4} (C_{lp})_w (C_{np})_w \frac{\phi}{C_l} \left(\frac{\phi}{C_n} \right)^* \right]$ is the cross-power term between the rolling and yawing moments on the wing, and had the coefficient $(C_{rp})_w$ not been zero two other cross-power terms would have appeared.

The expressions of equations (D8) and (D11) are equivalent. Either form may be used, but, for the purposes of part II of this report, equation (D8) is more useful and the illustrative tabulation process has been set up on this basis.

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